Breakdown of Alfvén's Theorem in Ideal MHD

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Conservation of magnetic flux in a perfectly conducting fluid is one of the most fundamental conservation laws of magnetohydrodynamics (MHD). This result is also called *Alfvén's Theorem*, after Hannes Alfvén, who was the first to state it in a foundational paper on MHD in 1942. Intuitively, this theorem implies that magnetic-field lines are 'frozen' into the fluid, so that the magnetic-field lines and the plasma move together. It is an analogue of the Kelvin-Helmholtz Theorem in ideal hydrodynamics, where there is conservation of circulation and lines of vorticity are 'frozen' to the inviscid fluid. One important consequence of this theorem is that the topology of magnetic-field lines is preserved, and, in particular, that crossing field lines cannot reconnect. This presents a puzzle in many situations where ideal MHD is supposed to hold to a very good (leading-order) approximation—such as in many astrophysical systems at very high magnetic Reynolds numbers—and yet magnetic-field lines are observed to reconnect all the time.

Alfvén's Theorem rests, however, on certain implicit assumptions about the smoothness of the velocity and magnetic fields, which can be violated in high Reynolds number turbulent plasmas. We show in [2] that violation of magnetic flux conservation is possible due to non-linear effects, if the MHD solutions are sufficiently singular. In the spirit of renormalization-group methodology, we analyze an effective equation for MHD modes at length-scales $> \ell$, with smaller scales eliminated. We prove that flux conservation can be violated at an instant of time for an arbitrary ℓ, including

$\ell \to 0$, provided that at least one of three necessary conditions is satisfied. These conditions are (i) non-rectifiability of advected loops, (ii) unbounded velocity or magnetic fields, and/or (iii) singular current and vortex sheets that both exist and intersect in a set of large enough dimension. Mathematically, our theorem is analogous to Onsager's result on the energy dissipation anomaly in hydrodynamic turbulence and mirrors the breakdown of the Kelvin-Helmholtz theorem proved in [3].

We have recently extended this analysis to take into account the time dimension, establishing necessary conditions for the breakdown of flux conservation over a finite interval of time. Our conclusions are illustrated by an exact solution of 'adiabatic MHD' found by Harold Grad [4] which exhibits ideal magnetic reconnection. We also present results of a numerical simulation of the full MHD equations at a high magnetic Reynolds number, which confirm our theoretical picture.

References

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