Chaotic motion of the ring configuration of N-vortex points on sphere

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Abstract submitted to EE250

We consider the evolution of the vortex points on a sphere, which is often used to describe geophysical flows on Earth. The governing equations are derived from the two-dimensional incompressible Euler equations on the sphere by assuming that the vorticity is concentrated in points discretely. Specifically, the motion of the vortex point with the strength Γ_m, located at (Θ_m, Ψ_m) in the spherical coordinates, is given by

$$\dot{\Theta}_m = -\frac{1}{4\pi}\sum_{j\neq m}^{N}\frac{\Gamma_j \sin\Theta_j \sin(\Psi_m - \Psi_j)}{1-\cos\Theta_m\cos\Theta_j - \sin\Theta_m\sin\Theta_j\cos(\Psi_m-\Psi_j)}, \tag{1}$$

$$\begin{aligned}\dot{\Psi}_m &= -\frac{1}{4\pi\sin\Theta_m}\sum_{j\neq m}^{N}\frac{\Gamma_j[\cos\Theta_m\sin\Theta_j\cos(\Psi_m-\Psi_j)-\sin\Theta_m\cos\Theta_j]}{1-\cos\Theta_m\cos\Theta_j-\sin\Theta_m\sin\Theta_j\cos(\Psi_m-\Psi_j)} \\ &\quad +\frac{\Gamma_n}{4\pi}\frac{1}{1-\cos\Theta_m}-\frac{\Gamma_s}{4\pi}\frac{1}{1+\cos\Theta_m}, \qquad m=1,2,\cdots,N,\end{aligned} \tag{2}$$

where Γ_n and Γ_s denote the strengths of the vortex points fixed at the north and the south poles of the sphere. They are introduced to incorporate with an effect of rotation of the sphere. An important property of the equations is that they are rewritten in a Hamiltonian form[1], to which we can apply the abstract theory of the Hamiltonian dynamical systems.

Now, we focus on the polygonal ring configuration of the identical N vortex points equally spaced along the line of latitude θ_0, called N-ring. The configuration is of significance since such coherent vortex structure is often observed in many planetary flows. The N-ring is a relative fixed configuration of (1) and (2) rotating with a constant longitudinal speed. The linear stability analysis of the N-ring[2] shows that there exist double zero eigenvalues and the other eigenvalues, $\lambda_m^\pm$ for $m = 1, \cdots, N-1$, satisfy $\lambda_m^\pm = \lambda_{N-m}^\pm$. Hence, the eigenvalues $\lambda_M^\pm$ are simple and $\lambda_m^\pm$ $(m = 1, \cdots, M-1)$ are double for $N = 2M$, whereas all the eigenvalues $\lambda_m^\pm$ $(m = 1, \cdots, M)$ are double for $N = 2M + 1$. Moreover, since $\left(\lambda_i^\pm\right)^2 < \left(\lambda_j^\pm\right)^2$ holds for $1 \leq i < j \leq M$, the stability of the N-ring is determined by that of the largest eigenvalues $\lambda_M^\pm$.

The present study deals with how the N-ring evolves when it becomes unstable. When the largest eigenvalue λ_M^+ is unstable and the others still remain neutrally stable, it is shown that the unstable and stable manifolds corresponding to $\lambda_M^\pm$ have the saddle connection[2]. Moreover, when we reduce the equations to low-dimensional Hamiltonian systems with a certain invariant property in terms of some discrete transformations on the sphere[3], a general theory of the Hamiltonian system with saddle centers based on the Melnikov-type analysis[4, 5] is applicable to the reduced systems. As a consequence of the analysis, we show that the unstable N-ring exhibits a horseshoe-like chaotic behavior.

References

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