# Local Geometric Properties and Global Regularity of Incompressible Inviscid Flows


Jian Deng[1], Thomas Y. Hou[2], Ruo Li[3] and Xinwei Yu[4]

1. School of Mathematical Sciences, Fudan University, China
2. Applied and Computational Mathematics, Caltech, USA
3. School of Mathematical Sciences, Peking University, China
4. Mathematics Department, UCLA, USA



*December 2, 2006*


Whether incompressible flows governed by the unforced 3D Navier-Stokes equations

$$\begin{aligned} \boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} &= -\nabla p + \nu \triangle \boldsymbol{u} \\ \nabla \cdot \boldsymbol{u} &= 0 \end{aligned}$$

($\nu > 0$) or the 3D Euler equations ($\nu = 0$), starting from smooth initial data, will remain smooth for all time or form a singularity in finite time is a long-standing open problem of great interest to both mathematical and physical communities ([Con97], [Fri95]).

It has been shown (Constantin [Con86]) that as long as the solution to the Euler equations stays smooth, the solution to the Navier-Stokes equations with the same initial data will remain smooth for arbitrarily long time when the viscosity is small enough. Thus to solve both singularity problems it is important to understand the dynamics of the Euler flow.

The dynamics of the Euler flow can be reduced to the dynamics of curves called vortex lines that remain for all time tangent to the vorticity vector field $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ everywhere. Vortex lines are not only transported, but also twisted and stretched by the flow. Since the vorticity magnitude $|\boldsymbol{\omega}|$ is just the local stretching of vortex lines, in light of the famous result by Beale, Kato and Majda ([BKM84]), which states that

$$\int_0^T \max |\boldsymbol{\omega}(\,\cdot\,, t)| \, \mathrm{d}t = \infty$$

is a necessary and sufficient condition for the formation of a finite time singularity at time $T$, it is important to understand the dynamics of vortex lines in the context of possible finite time singularity formation.

In Deng-Hou-Yu [DHY05], [DHY06], subtle relations between the twisting of vortex lines and the possible formation of finite time singularity are revealed. More specifically, a small segment of a vortex line in the high vorticity region is considered. Let its length be denoted by $L(t)$, the maximum curvature along it by $K(t)$, the maximum $\left|\nabla \frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|}\right|$ along it by $M(t)$, and the maximum tangential and normal velocities along it by $U_{\boldsymbol{\xi}}(t)$, $U_{\boldsymbol{n}}(t)$ respectively. Then the following was shown.

Let $T$ be any given time. The local stretching stays finite up to $T$, that is no finite time singularity can form up to $T$, whenever the following conditions are satisfied.

1. $U_{\boldsymbol{\xi}}(t) + K(t)\, L(t)\, U_{\boldsymbol{n}}(t) = O\Big( (T-t)^{-A} \Big)$ for some $A < 1$
2. $K(t)\, L(t), M(t)\, L(t) = O(1)$, and
3. $L(t)^{-1} = O\Big( (T-t)^{-B} \Big)$ for $B < 1 - A$.

Thus vortex lines have to be severely twisted for finite time singularities to form.

The above conditions 1–3 are all verified by the recent numerical computation by Hou and Li [HL06] for the anti-parallel vortex tube initial data.

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