Anisotropy in TURBULENCE

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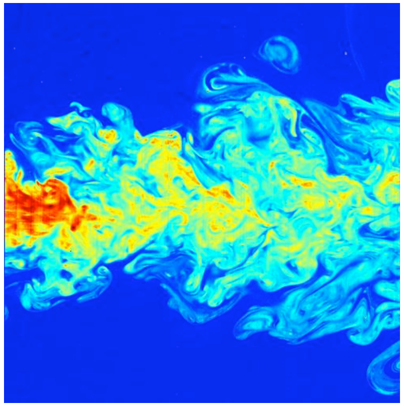
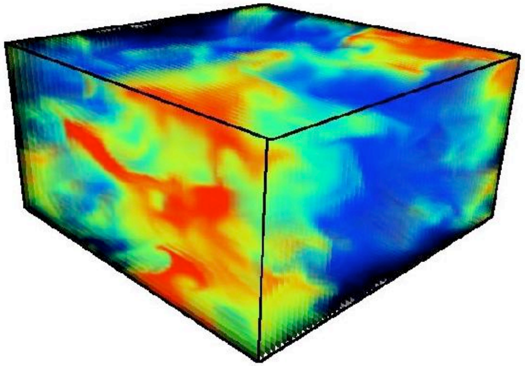
$$
\begin{cases}
\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\partial} \boldsymbol{v} = -\boldsymbol{\partial} P + \nu \partial^2 \boldsymbol{v} + \boldsymbol{f} \\
\boldsymbol{\partial} \cdot \boldsymbol{v} = 0 \\
+ \text{ boundary conditions}
\end{cases}
$$

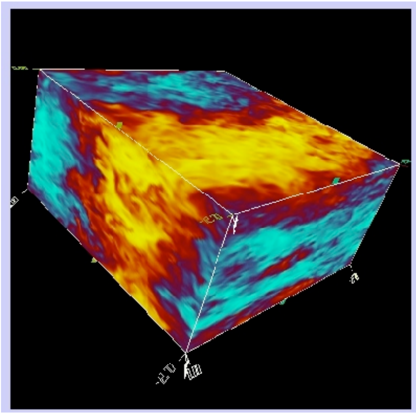
- Kinematics + Dissipation are invariant under Rotation+Translation
- Non-universal statistical behaviour <-> Anisotropy
- Small scales vs large scales



Turbulent jet



3d Convective Cell



Shear Flow

I. Arad, V. L'Vov I. Procaccia PRE 59, 6753 (1999).
Arad et al. PRL 82, 5040 (1999)
Arad et al. PRL 81, 5330 (1998).

$$S_n^{\alpha_1\cdots\alpha_n}(\boldsymbol{r}) \stackrel{\text{def}}{=} \langle \delta v^{\alpha_1}(\boldsymbol{x},\boldsymbol{r},t)\cdots\delta v^{\alpha_n}(\boldsymbol{x},\boldsymbol{r},t)\rangle \ ,$$
$$\delta\boldsymbol{v}(\boldsymbol{x},\boldsymbol{r},t) \stackrel{\text{def}}{=} \boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r},t)-\boldsymbol{v}(\boldsymbol{x},t) \ ,$$

3d rotation

$$x'_\alpha = \Lambda_{\alpha,\beta}x_\beta$$

Decomposition in terms of (irreducible) invariant subset -labelled by q,j=0,1,2,...

Set of 3n*(2j+1) Eigenfunctions of group of rotations in 3d: $B_{q,jm}^{\alpha_1\ldots\alpha_n}(\mathbf{r})$

n-rank tensor which depends on a 3d vector

$$S_n^{\alpha_1\cdots\alpha_n}(\boldsymbol{r}) = \sum_{j=0}\sum_{m=-j}\ \ldots_{jm}\ \ldots = \sum_{qjm}\mathcal{S}_{qjm}(r)B_{qjm}^{\alpha_1\cdots\alpha_n}(\hat{\boldsymbol{r}}) \ .$$

The simplest set of 0-rank tensor (SCALAR) observable:

Longitudinal Structure Functions

$$S^{(n)}(\boldsymbol{r}) = \langle [(\boldsymbol{v}(\mathrm{x}+\boldsymbol{r}) - \boldsymbol{v}(\mathrm{x})) \cdot \hat{\boldsymbol{r}}]^n \rangle \, .$$

$$S^{(n)}(\boldsymbol{r}) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \mathcal{S}_{jm}^{(n)}(r) Y_{jm}(\hat{\boldsymbol{r}}).$$

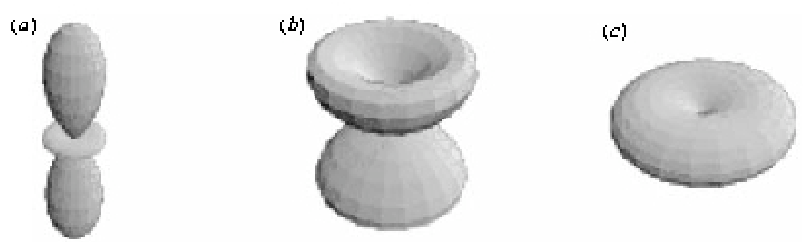


FIGURE 4. Graphical representation of spherical harmonics (a) $|Y^{2,0}(\theta,\phi)|$, (b) $|Y^{2,1}(\theta,\phi)|$, and (c) $|Y^{2,2}(\theta,\phi)|$.

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\partial v} = -\boldsymbol{\partial} P + \nu \partial^2 \boldsymbol{v} + \boldsymbol{f}$$

$$\partial_t v_i + \Gamma_{ijk}(v_j v_k) - \nu \Delta v_i = f_i$$

$$\partial_t S^n + \Gamma^{n+1} S^{n+1} - \nu D^n S^n = \langle \delta f_1 \delta v_2 \cdots \delta v_{n-1} \rangle + perm.$$

rotational invariant operator

$r << L_f$

$$\partial_t S^n + \Gamma^{n+1} S^{n+1} - \nu D^n S^n \sim 0$$

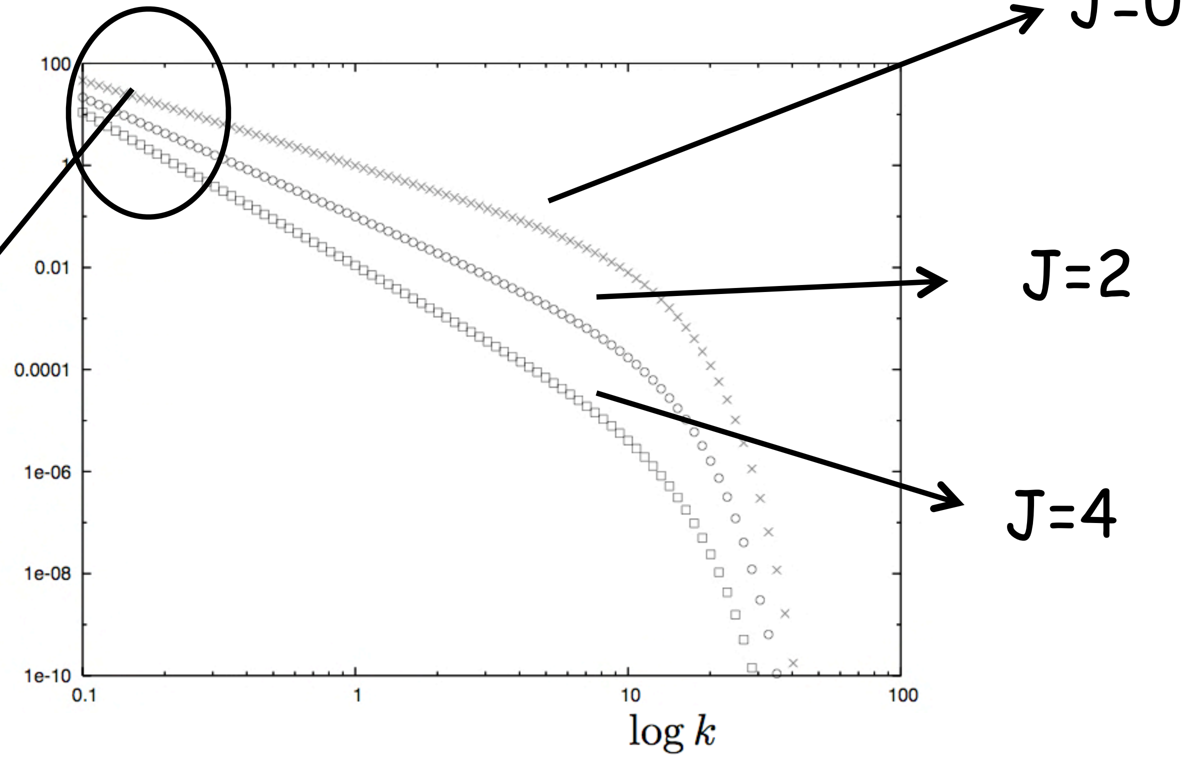
+ so(3) -> $S^{(n)}_{\alpha_1 \ldots \alpha_n}(\mathbf{r}) = \sum_{jmq} \mathcal{S}^{(n)}_{jmq}(r) B^{jmq}_{\alpha_1 \ldots \alpha_n}(\hat{r})$

$$\partial_t \mathcal{S}^n_{jmq} + \sum_{q'} \Gamma^{n+1}_{jmq'} \mathcal{S}^{n+1}_{jmq'} - \nu D^n_{jmq} \mathcal{S}^n_{jmq} = 0$$

FOLIATION !!!

$$\partial_t \mathcal{S}^{(n)}_{jq} + \sum_{q'} \Gamma^{(n+1)}_{jq'} \mathcal{S}^{(n+1)}_{jq'} - \nu D^{(n)}_{jq} \mathcal{S}^{(n)}_{jq} \sim 0$$

$\log E^{(j)}(k)$



Large scale physics:
all sectors coupled by
forcing terms

$\nu \to 0$ $$\partial_t \mathcal{S}^{(n)}_{jq} + \sum_{q'} \Gamma^{(n+1)}_{jq'} \mathcal{S}^{(n+1)}_{jq'} - \nu D^{(n)}_{jq} \mathcal{S}^{(n)}_{jq} \sim 0$$

$$\mathcal{S}^{(n)}_{jmq}(r) \propto a_{jmq}\left(\frac{r}{L}\right)^{\zeta^j_n}$$ scaling ?

$$S^{(n)}_{jqm}(r) \sim A_{jmq}(\frac{r}{L})^{\xi^j_n}$$

Working Hypothesis

•projection on each sector has a **universal** scaling exponent, depending on that sector **only**.

•Dependency on large scale physics shows up only in **prefactors**

•Pure power laws **only** in each separeted sector:

$$S^{(n)}(\mathbf{r}) \sim \sum_j A_j(\frac{r}{L})^{\zeta^j_n} \longrightarrow S^{(n)}(\mathbf{r}) \sim A_0(\frac{r}{L})^{\zeta^0_n} + A_1(\frac{r}{L})^{\zeta^1_n} + \cdots$$

$$S^{(n)}(\mathbf{r}) \sim A_0(\frac{r}{L})^{\zeta_n^0} + A_1(\frac{r}{L})^{\zeta_n^1} + \cdots$$

- Matching Infra-Red boundary conditions: $r \sim L$

$$S^{(n)}(\mathbf{L}) \sim A_0 + A_1 + A_2 + \cdots$$

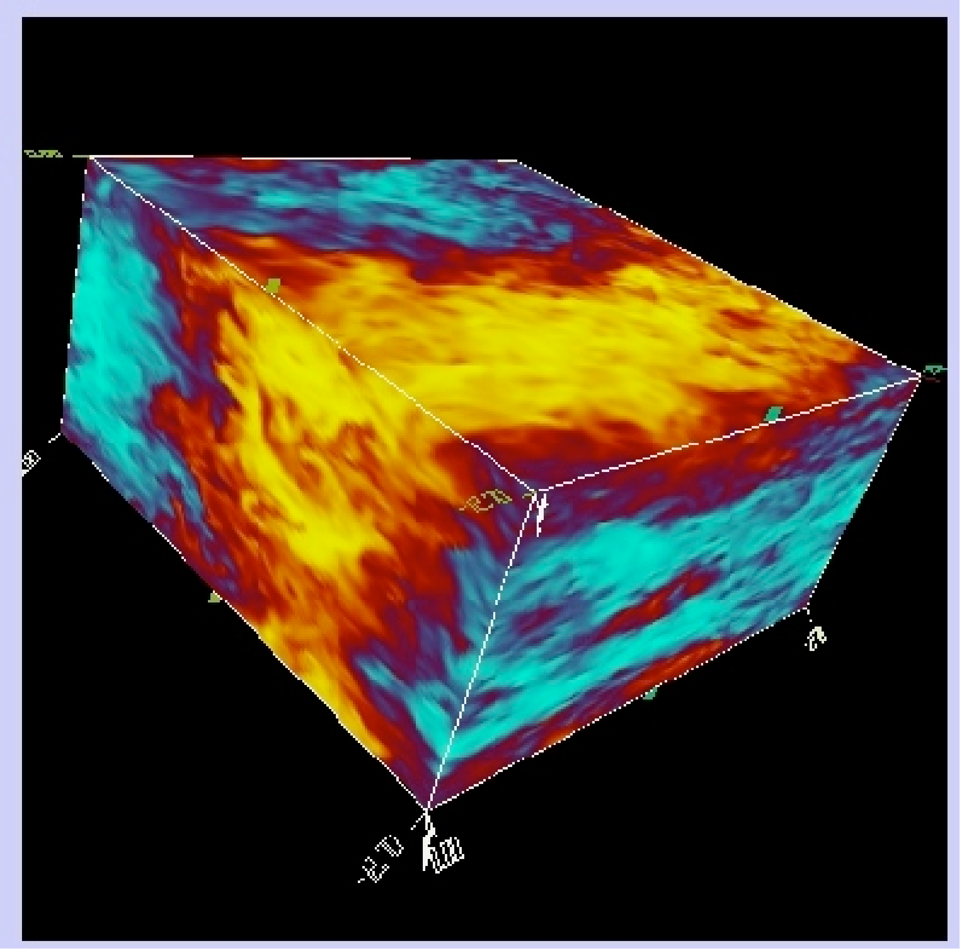
prefactor <u>cannot be universal</u>

- About universality of scaling exponents nothing can be said rigorously, at least for the NS eqs.

- Recovery of Isotropy
- Small-Scales Universality

$$\zeta^{j=0}(n) \le \zeta^{j=1}(n) \le \zeta^{j=2}(n) < \ldots$$

Scaling in anisotropic sectors



We performed a DNS of a Random-Kolmogorov Flow

- Periodic boundary conditions
- 256x256x256
- Hyperviscosity
- Homogeneous but Anisotropic

$$f_z = cos(z + \phi(t))$$

$$\langle \phi(t)\phi(t') \rangle = \delta(t - t')$$

L.B. and F. Toschi, PRL 86, 4831 (2001)

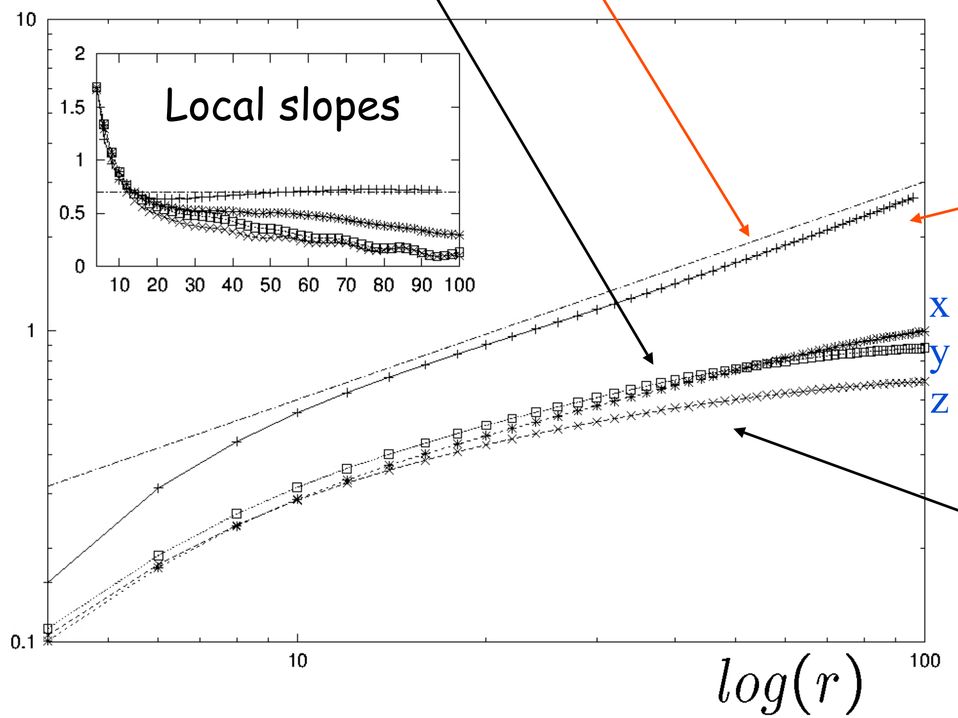
L.B. I. Daumont, A. Lanotte and F. Toschi. PRE. 66, 056306 (2002)

Comparison of scaling properties:
isotropic sector (j=0,m=0) vs undecomposed structure function

$$S_n(\mathbf{r}) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} S_n^{jm}(|r|) Y_{jm}(\hat{r})$$

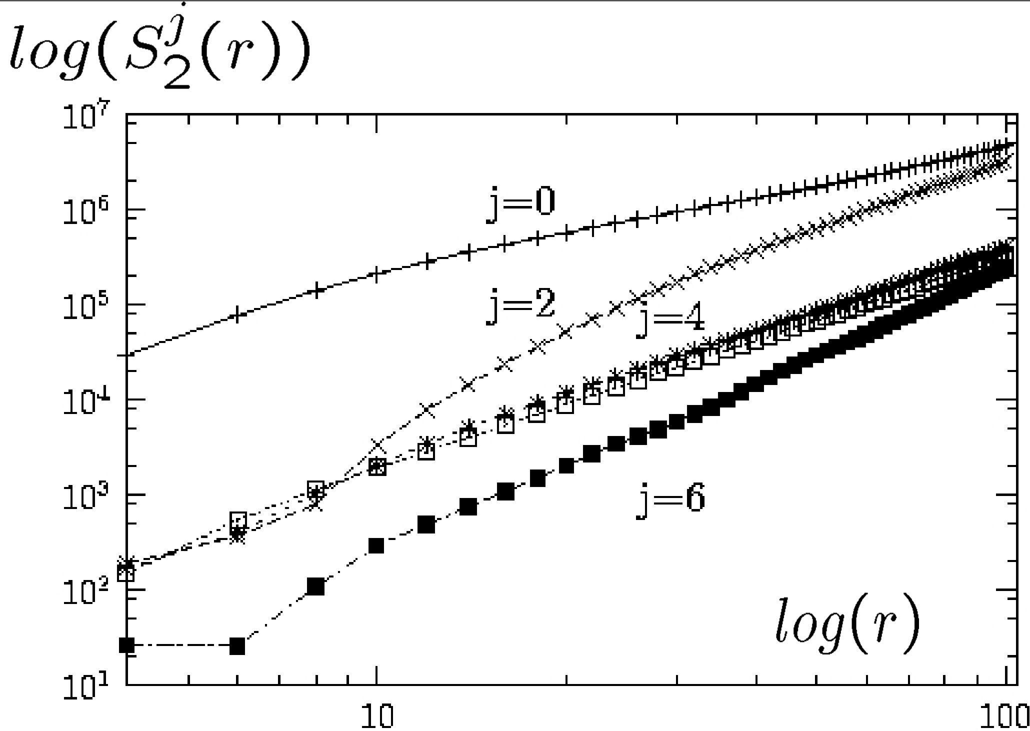
$$S_n(\mathbf{r}) = a_0 r^{\zeta_n^{j=0}} + a_2 r^{\zeta_n^{j=2}} + a_4 r^{\zeta_n^{j=4}} + \cdots$$

log (2nd order structure function)



isotropic sector

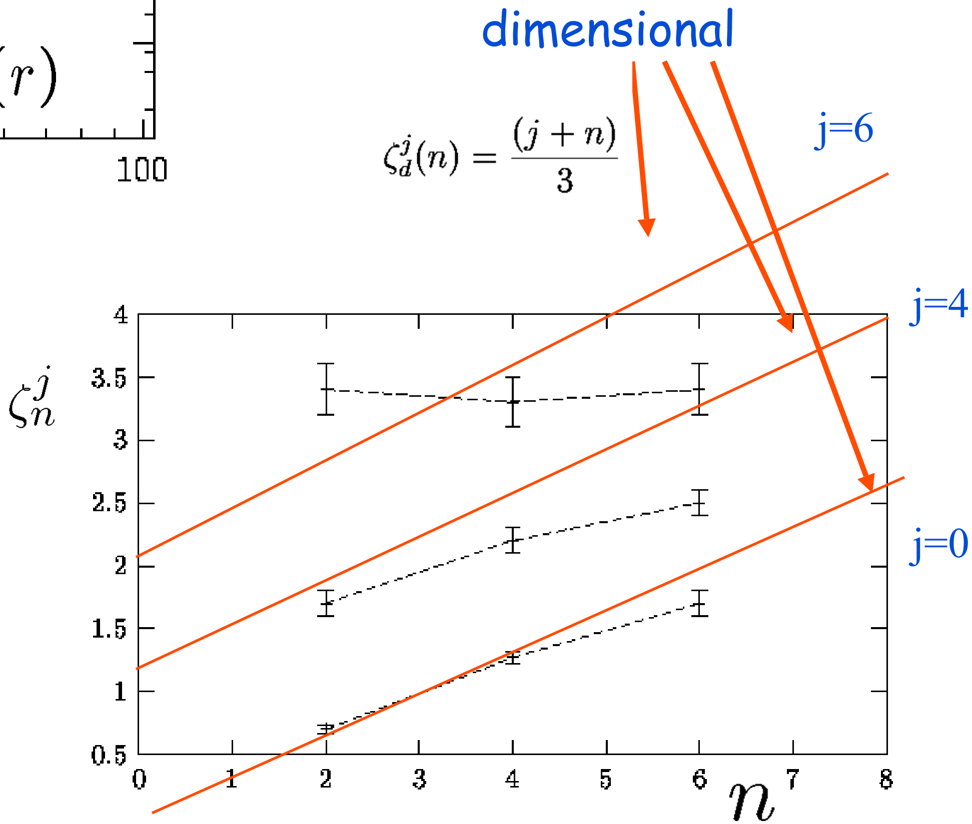
before so(3) decomposition



scaling is m-independent

$$\zeta_n^{j=0} < \zeta_n^{j=1} < \zeta_n^{j=2} < \cdots$$

Recovery of isotropy

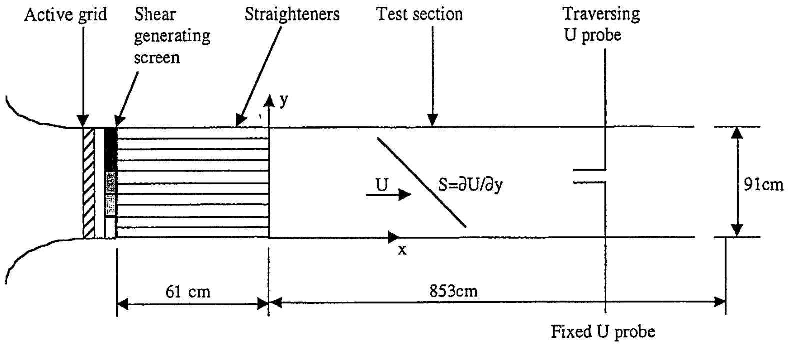


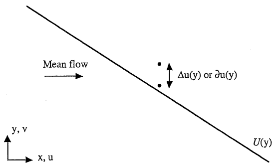
L.B. I. Daumont, A> Lanotte and F. Toschi
PRE. 66, 056306 (2002)

Recovery of isotropy vs persistency of Anisotropies

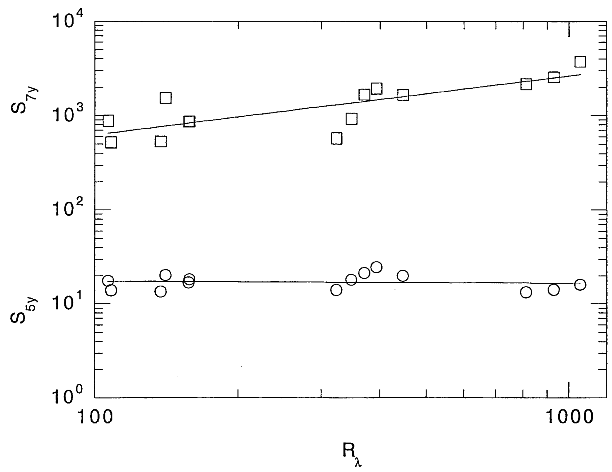
Experimental Results on Persistency of Anisotropies

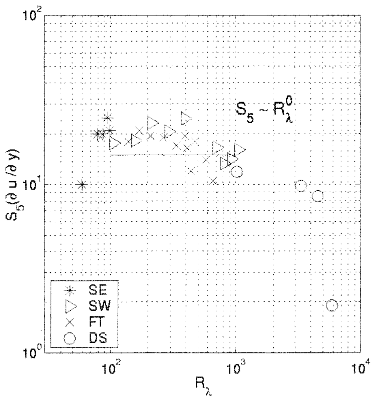
Garg and Warhaft, PoF 10, 662 (1998).
Kurien et al. PRE 61, 407 (2000).
Kurien and Sreenivasan, PRE 62, 2206 (2000).
Shen and Warhaft, PoF 14, 370 and 2432 (2002).



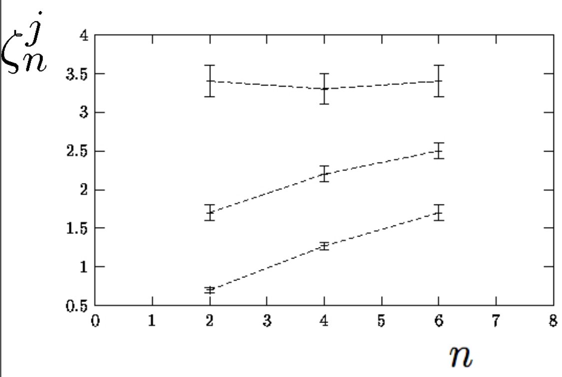


$$S^{(2n+1)}(R_\lambda) = \frac{\langle(\partial_y v_x)^{2n+1}\rangle}{\langle(\partial_y v_x)^2\rangle^{\frac{2n+1}{2}}}. = \frac{\text{anis.}}{\text{iso}}$$





$$S^{(n)}(r,\hat{r}) = A_0 r^{\zeta_n^{j=0}} + A_2(\hat{r}) r^{\zeta_n^{j=2}} + A_4(\hat{r}) r^{\zeta_n^{j=4}} + \cdots$$



$$\zeta_n^{j=0} \leq \zeta_n^{j=1} \leq \zeta_n^{j=2} < \ldots$$

<u>Two ways to measure small-scales anisotropies:</u>

aniso(n)/iso(n)

1)

$$F_j^{(n)}(r) = \frac{\mathcal{S}_j^{(n)}(r)}{\mathcal{S}_{j=0}^{(n)}(r)} \sim r^{\Delta_n^j} \to 0; \quad \Delta_n^j = \zeta_n^j - \zeta_n^0 > 0$$

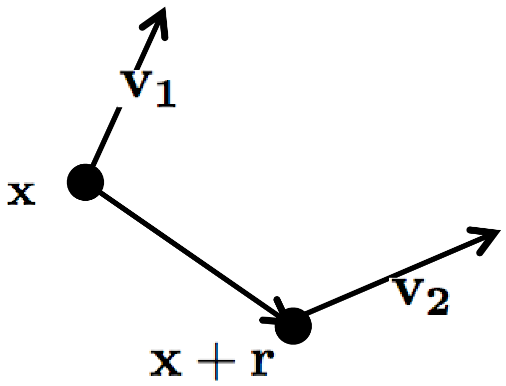
aniso(n)/iso(n=2)

2)

$$K_j^{(n)}(r) = \frac{\mathcal{S}_j^{(n)}(r)}{(\mathcal{S}_{j=0}^{(2)}(r))^{\frac{n}{2}}} \sim r^{\Delta_n^j} \to ?; \quad \Delta_j^n = \zeta_n^j - \frac{n}{2}\zeta_2^0 <=> 0$$

L.B. and M. Vergassola, PoF. 13, 2139 (2001)

Open questions



$$\delta_L v(r) = (\mathbf{v_1} - \mathbf{v_2}) \cdot \mathbf{r}$$

$$\delta_T v(r) = (\mathbf{v_1^\perp} - \mathbf{v_2^\perp})$$

<u>fully isotropic</u>

n=2
J=0

$$\begin{cases} \langle \delta v^\alpha(\mathbf{r}) \delta v^\beta(\mathbf{r}) \rangle = a(r)\hat{r}^\alpha \hat{r}^\beta + b(r)\delta^{\alpha\beta} \\ S_L^{(2)}(r) = \langle (\delta_L v)^2 \rangle = a(r) + b(r) \\ S_T^{(2)}(r) = \langle (\delta_T v)^2 \rangle = b(r) \end{cases}$$

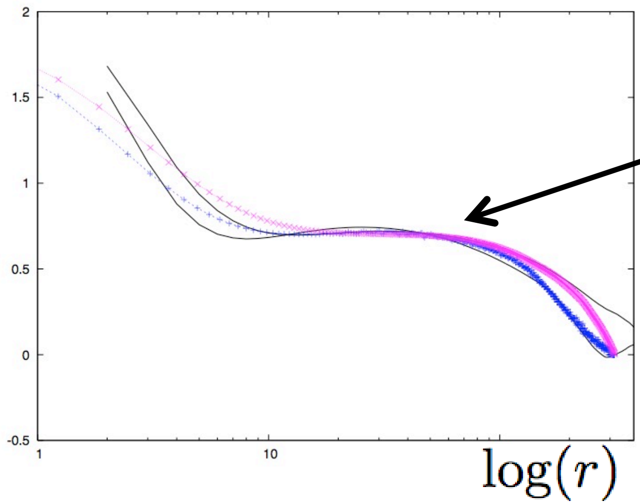
n=4
J=0

$$\begin{cases} \langle \delta v^\alpha \delta v^\beta \delta v^\gamma \delta v^\delta \rangle \sim c(r)\hat{r}^\alpha \hat{r}^\beta \hat{r}^\gamma \hat{r}^\delta + d(r)[\hat{r}^\alpha \hat{r}^\beta \delta^{\gamma\delta} + perm] + e(r)[\delta^{\gamma\delta}\delta^{\alpha\beta} + perm] \\ S_L^{(4)}(r) = \langle (\delta_L v)^4 \rangle = c(r) + 3d(r) + 3e(r) \\ S_T^{(4)}(r) = \langle (\delta_T v)^4 \rangle = 3e(r) \\ S_{LT}^{(4)}(r) = \langle (\delta_T v)^2 (\delta_L v)^2 \rangle = 3d(r) + 3e(r) \end{cases}$$

1024^3, T. Gotoh, D. Fukayama and T. Nakano, PoF 2002
2048^3, R. Benzi, LB, R. Fisher, L. Kadanoff, D. Lamb and F. Toschi, unpub 2007

$$\xi_T(2) = \frac{d\, log S_T^{(2)}}{d \log r}$$

$$\xi_L(2) = \frac{d\, log S_L^{(2)}}{d \log r}$$

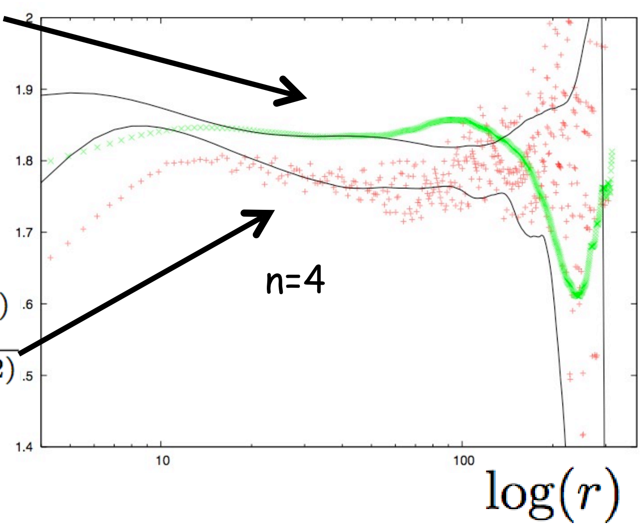


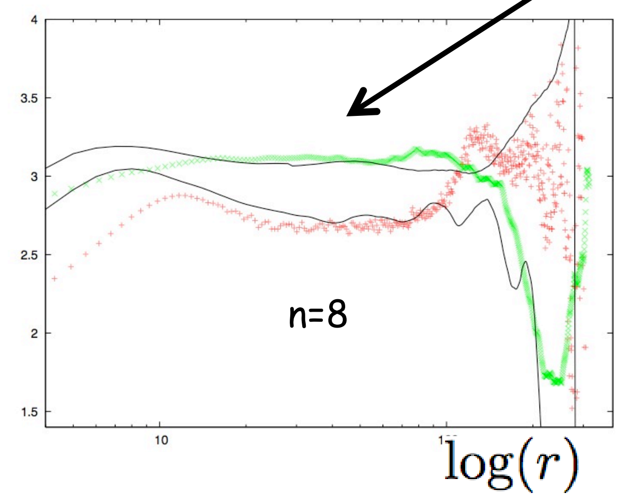
Exact: equal if isotropic and incompressible (good)

real? $Re \to \infty$?

$$\frac{\xi_L(4)}{\xi_L(2)} = \frac{d\, log S_L^{(4)}}{d \log S_L^{(2)}}$$

$$\frac{\xi_T(4)}{\xi_T(2)} = \frac{d\, log S_T^{(4)}}{d \log S_T^{(2)}}$$





Conclusions

- SO(3) decomposition is needed if you want to disentangle in a systematic way isotropic from anisotropic contributions and different anisotropic contributions among themselves.

- Dynamical importance through the "foliation" mechanism of the eqs. of motion.

- (i) Power law behaviour only in separated (j) sectors; (ii) intermittency also in anisotropic sectors, (iii) (slow) Recovery of small-scales isotropy.

- OPEN QUESTIONS: (i) Universality of anisotropic exponents? (ii) longitudinal vs transverse scaling in isotropic sector.

Credits: I. Arad, G. Boffetta, A. Celani, I. Daumont, A. Lanotte, D. Lohse, I. Mazzitelli, I. Procaccia, F.Toschi, M. Vergassola

For a recent review see:

L. Biferale & I. Procaccia
Anisotropy in turbulent flows and in turbulent transport
Physics Reports Volume 414, Issues 2-3 , July 2005, Pages 43-164