

GENERAL LAWS OF THE MOTION OF FLUIDS¹

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1. Having established in my previous Memoir² the principles of fluid equilibrium in their most general form, in terms of both the diverse qualities of fluids and the forces that act upon them, I now propose to deal with the motion of fluids in the same way and to seek out the general principles on which the entire science of fluid motion is based. It will readily be understood that this is a much more difficult undertaking and involves studies of incomparably greater depth. Nevertheless, I hope to arrive at an equally successful conclusion, so that, if difficulties remain, they will pertain not to Mechanics but purely to Analysis, this science not yet having been brought to the degree of perfection necessary to develop analytical equations (formules) that embody the principles of fluid motion.

2. The task, then, is to discover the principles by means of which the motion of a fluid can be determined, whatever its state and whatever the forces to which it is subjected. To this end, we shall examine in detail all the elements which form the subject of our research and contain quantities both known and unknown. First of all, the nature of the fluid is assumed to be known, in which case it is necessary to consider its various forms since it may be compressible or incompressible. If it is not compressible, then there are two possibilities: either the entire mass is composed of homogeneous parts, whose density is everywhere and always the same, or it is composed of heterogeneous parts and in this case it is necessary to know the density of each component and the proportions of the mixture. If the fluid is compressible and its density is variable, we must know the law according to which its elasticity³ depends on the density and whether the elasticity depends only on the density or also on some other property, such as heat,⁴ which is proper to each particle of fluid, at least for each instant of time.

3. It must also be assumed that the state of the fluid at a certain moment of time is known and I shall call this the initial state ("état primitif") of the fluid. As this state is quasi-arbitrary, it is necessary, first of all, to know the distribution of the particles of which the fluid is composed and, unless in the initial state the fluid is at rest, the motion impressed upon them. However, the initial motion is not entirely arbitrary since both the continuity and the impenetrability of the fluid impose a certain limitation which I shall investigate below. Often, however, nothing is known of the initial state, for example when it is a question of determining the motion of a river, and then it is usually only possible to seek the steady state at which the fluid finally arrives, thereafter undergoing no further changes. Now, neither this circumstance nor the initial state in any way affect the investigation to be made and the calculations will always be the same. It is only in the integrations that they need to be taken into account for the purpose of determining the constants which every integration involves.

¹ This is an English translation by Thomas E. Burton of Euler's memoir "Principes généraux du mouvement des fluides" (*Mémoires de l'Académie Royale des Sciences et Belles Lettres*, année 1755. Berlin, 1757. v. 11, pp. 274-315 = *Opera Omnia*, ser. II, v. 12, pp. 54-91). The original manuscript of this memoir has not been preserved and therefore it is not known whether Euler first wrote it, as usual, in Latin or directly in French. In any event, the French of the memoir retains traces of Latinisms. It should be borne in mind that in the XVIIIth century the language of scientific literature was not yet as formalized as it is today and therefore some of the colorfulness of the original must inevitably be lost in translation.

A detailed critique of Euler's published work can be found in Truesdell's preface to the corresponding volume of L. Euleri *Opera Omnia*, ser. II, v. 12, pp. LXXXIV-XCI. The article by G. K. Mikhailov (see this issue), who, together with G. Yu. Stepanov, has provided the footnotes, offers some general historical and scientific comments on Euler's work.

In reading Euler's mathematical formulas, it should be kept in mind that he employed a system of physical units different from that in use today. Explanatory footnotes have been supplied where necessary (see also the above-mentioned article by G. K. Mikhailov).

² L. Euler, "Principes généraux de l'état d'équilibre des fluides", *Mémoires de l'Académie des sciences et belles lettres*, année 1755. Berlin, 1757. v. 11, pp. 217-273 = *Opera Omnia*, ser. II, v. 12, pp. 2-53).

³ By elasticity ("élasticité") Euler means that property of a fluid which is expressed in the creation of internal pressure and therefore uses the term on an equal footing with the term "pressure" (see §5 below).

⁴ Essentially, heat ("chaleur") should be taken to mean temperature.

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4. Thirdly, the data must include the external forces to which the fluid is subjected. I shall call these forces external to distinguish them from the internal forces which the fluid particles exert on each other and which will constitute the main topic for subsequent investigation. Thus, it could be assumed that the fluid is not exposed to any external force, unless it be natural gravity which is everywhere considered to be constant in magnitude and to act in the same direction. However, to generalize the investigation, I shall consider the fluid to be acted upon by forces which may be directed towards one or more centers or obey some other law with respect to both magnitude and direction. As far as these forces are concerned, only their accelerating action is directly known, irrespective of the masses upon which they act. Accordingly, I shall introduce into the calculations only the accelerative forces, from which it will be easy to obtain the true motive forces by multiplying in each case the accelerative forces by the masses to which they are applied.⁵

5. Let us now turn to those elements which contain that which is unknown. In order properly to understand the motion that will be imparted to the fluid it is necessary to determine, for each instant and for each point, both the motion and the pressure ("pression") of the fluid situated there. And if the fluid is compressible, it is also necessary to determine the density, knowing the above-mentioned other property which, together with the density, makes it possible to determine the elasticity. The latter, being counterbalanced by the fluid pressure, must be considered equal to that pressure, exactly as in the case of equilibrium, where I have developed these ideas more thoroughly.⁶ Clearly, then, the number of quantities which enter into the study of fluid motion is much greater than in the case of equilibrium, since it is necessary to introduce letters which denote the motion of each particle and all these quantities may vary with time. Thus, in addition to the letters which determine the location of each conceivable point in the fluid, another is required which denotes the time already elapsed and which, by virtue of its variability, can be applied to any given time.

6. Suppose (Fig. 1) that from the initial state a time t has elapsed and that the fluid is now in a state of motion which is to be determined. Whatever the volume that the fluid now occupies, I begin by considering some point Z in the fluid mass and in order to introduce the location of this point Z into the calculations I relate it to three fixed axes, OA , OB and OC , mutually perpendicular at the point O and having a given position. Let the two axes OA and OB lie in the plane represented by the page and let the third OC be perpendicular to it. Then from the point Z we draw a perpendicular ZY to the plane AOB and from the point Y a normal YX to the axis OA to obtain three coordinates: $OX=x$, $XY=y$ and $YZ=z$ parallel to our three axes. For each point in the fluid mass, these three coordinates x , y and z will have specific values and by successively giving these three coordinates all possible values, both positive and negative, we can run through all the points of infinite space, including those lying in the volume occupied by the fluid at each instant of time.

7. Secondly, I shall consider the accelerative forces which act at a given moment on the fluid particle located at Z . Now, whatever these forces may be, they can always be reduced to three acting in the three directions ZP , ZQ and ZR parallel to our three axes OA , OB and OC . Taking the accelerative force of natural gravity⁷ as the unit, we let P , Q and R be the accelerative forces acting on the point Z in the directions ZP , ZQ and ZR , the letters P , Q and R denoting abstract numbers ("nombres absolus").⁸ If the same forces always act at the same point in space Z , the quantities P , Q and R will be expressed by certain functions of the three coordinates x , y and z . However, if the forces also vary with time t , these functions will likewise contain time t . I shall assume that these functions are known, since the acting forces must be included among the known quantities, whether they depend only on the variables x , y , z or also on time t .

8. Let r now express the heat at the point Z or that other property which, in addition to the density, influences the elasticity in the case of a compressible fluid. The quantity r must also be considered to be a function of the three variables x , y , z and time t , since it might vary with time t at the same point Z in space. Thus, this function may be regarded as being known.⁹ Moreover, let the present density of the fluid particle located at Z be equal to q . As the unit of density I shall take the density of a certain homogeneous substance which I shall use to measure pressures in terms

⁵ Newton distinguishes between the "accelerative" and "motive" aspects of a force, the former being "a measure proportional to the velocity which it generates" and the latter "a measure proportional to the quantity of motion which it generates in a given time". Thus, the "accelerative force" is the ratio of the acting force to the mass of the particle on which it acts, i.e. the acceleration which it imparts, and the "motive force" is that which, strictly speaking, we now understand by force. The neutral term "acting" forces ("forces sollicitantes"), not used by Newton, was widely employed by Euler, starting with his well-known "Mechanics" (*Mechanica Sive Motus Scientia Analytice Exposita*. Vols. 1-2. Petropoli (= St Petersburg) (1736) = *Opera Omnia*, ser. II. V. 1).

⁶ See the Euler memoir mentioned in footnote 2.

⁷ The acceleration of gravity is intended.

⁸ The non-dimensionality of the values of P , Q and R is emphasized.

⁹ Euler is confining himself to the consideration of fluid motion in a given temperature field.

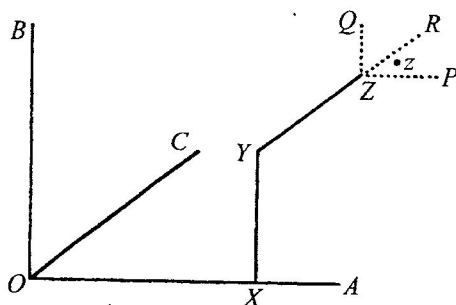


Fig. 1

of heights, as explained at greater length in my memoir on the equilibrium of fluids.¹⁰ Let, moreover, the present value of the fluid pressure at the point Z , expressed in terms of height, be equal to p , which will also denote the elasticity. Since the nature of the fluid is assumed to be known, we will know the relation between the height p and the quantities q and r .¹¹ Thus, p and q will likewise be functions, albeit unknown, of the four variables x, y, z and t . If the fluid is not compressible, the pressure p will be independent of the density q and the other property r will not enter into consideration at all.

9. Finally, whatever the motion corresponding at a given time to the fluid element located at the point Z , it too can be decomposed in the directions ZP, ZQ and ZR parallel to our three axes. Thus, let u, v and w be the velocities of this motion decomposed in the three directions ZP, ZQ and ZR . It is then obvious that these three quantities must also be considered to be functions of the four variables x, y, z and t . If the properties of these functions have been found and if the time t is assumed to be constant, then from the variability of the coordinates x, y and z the three velocities u, v and w and hence the true motion imparted to each element of the fluid at a given time will be known. If, however, the coordinates x, y and z are assumed to be constant and only the time t is considered to be variable, we shall find the motion not of some particular element of the fluid but of all the elements that pass successively through the same point Z ; in other words, at each moment of time the motion of that fluid element which is then located at the point Z will be known.

10. Let us consider what path will be described by a fluid element now at Z during the infinitely small time dt or the point at which it will be an instant later.¹² If we express the path as the product of velocity and time, a fluid element currently at Z will travel a distance $u dt$ in the direction ZP , a distance $v dt$ in the direction ZQ and a distance $w dt$ in the direction ZR . Therefore, if we set $ZP = u dt, ZQ = v dt$ and $ZR = w dt$ and from these three sides complete the construction of the parallelepiped, then the corner opposite the point Z will represent the point at which the fluid element in question will be after the time dt and the diagonal of the parallelepiped, which is equal to $dt(uu + vv + ww)^{1/2}$ will give the true path described.¹³ Consequently, the velocity of this true motion will be equal to $(uu + vv + ww)^{1/2}$ and the direction can easily be determined from the sides of the parallelepiped since it will be inclined to the plane AOB at an angle whose sine is equal to

$$\frac{w}{\sqrt{uu + vv + ww}}$$

to the plane AOC at an angle whose sine is equal to

¹⁰ Clearly, for Euler the density q is non-dimensional, being divided by the constant density ρ_0 of a certain auxiliary fluid: $q = \rho/\rho_0$. Euler defines the pressure in the fluid as the height p of a column of this same homogeneous auxiliary fluid. Thus, for Euler pressure is measured by a quantity with the dimension of length — the ratio of the acting pressure to the constant quantity $\rho_0 g$ (where g is the acceleration of gravity). For further details see the Euler memoir mentioned in footnote 2.

¹¹ That is, the "equation of state" of the moving medium is assumed to be known.

¹² The intuitive derivation of the equations of motion and continuity of an ideal (inviscid and non-heat-conducting) compressible fluid proposed by Euler is valid provided that the functions in question have bounded derivatives, up to and including the second. The modern derivation of these equations, based on the integral laws of conservation of mass and momentum of the fluid particles and the use of the Gauss theorem, is free of this limitation.

¹³ Euler usually writes xx rather than x^2 for the square of the quantity x .

$$\frac{v}{\sqrt{uu + vv + ww}}$$

and, finally, to the plane BOC at an angle whose sine is equal to

$$\frac{u}{\sqrt{uu + vv + ww}}$$

11. Having determined the motion of a fluid element which at a given instant is located at the point Z , let us now also examine that of some other infinitely close element located at the point z with the coordinates $x + dx$, $y + dy$ and $z + dz$. The three velocities of this element in the direction of the three axes can thus be expressed by u , v , w after substituting in those quantities $x + dx$, $y + dy$ and $z + dz$ or after adding to them their differentials while assuming time t to be constant. Thus, when $x + dx$ is substituted for x , the increments of u , v and w will be¹⁴:

$$dx \left(\frac{du}{dx} \right), \quad dx \left(\frac{dv}{dx} \right), \quad dx \left(\frac{dw}{dx} \right)$$

and when $y + dy$ is substituted for y , the increments will be:

$$dy \left(\frac{du}{dy} \right), \quad dy \left(\frac{dv}{dy} \right), \quad dy \left(\frac{dw}{dy} \right)$$

and the same will apply to the variation of z . Then, the three velocities of the fluid element currently located at z will be:

in the direction OA

$$u + dx \left(\frac{du}{dx} \right) + dy \left(\frac{du}{dy} \right) + dz \left(\frac{du}{dz} \right)$$

in the direction OB

$$v + dx \left(\frac{dv}{dx} \right) + dy \left(\frac{dv}{dy} \right) + dz \left(\frac{dv}{dz} \right)$$

in the direction OC

$$w + dx \left(\frac{dw}{dx} \right) + dy \left(\frac{dw}{dy} \right) + dz \left(\frac{dw}{dz} \right)$$

12. These are the velocities corresponding to a fluid element at the point z , which is infinitely close to the point Z and whose position is determined by the three coordinates $x + dx$, $y + dy$ and $z + dz$. Thus, if we choose a point z (Fig. 2) such that only x changes by dx , the other two coordinates y and z remaining the same as for the point Z , the three velocities of the fluid element located at this point z will be:

$$u + dx \left(\frac{du}{dx} \right), \quad v + dx \left(\frac{dv}{dx} \right), \quad w + dx \left(\frac{dw}{dx} \right)$$

These velocities will transport the element in the time dt to another point z' whose position must be determined relative to the point Z , namely the point to which the fluid element which was at Z is transported in the same time dt and whose position was determined above (see §10). For determining this point z' , I note that if the velocities of the point z were exactly the same as those of Z , then the point z' would fall at the point p ,¹⁵ so that the distance $Z'p$ would be equal and parallel to the distance Zz . Since, by hypothesis, Zz is parallel to the OA axis and equal to dx , the segment $Z'p$ will also be equal to dx and parallel to the OA axis.

13. Now, since the velocity along OA is not u but $u + dx(du/dx)$, this velocity increment will transport the element in question from p to q in the direction $Z'p$, so that

¹⁴ Rather than the now customary notation for partial derivatives using the symbol ∂ , Euler employs only the symbol d , but encloses the expressions for partial derivatives in round brackets.

¹⁵ Euler frequently uses the same notation for different quantities. Thus, both here and later on, the letters p and q , which in this article are mainly employed to denote pressure and density, are used to denote certain auxiliary points.

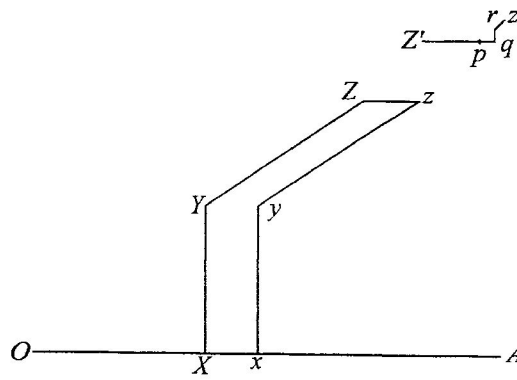


Fig. 2

$$pq = dt \, dx \left(\frac{du}{dx} \right)$$

Thus, this element would be at q , if the other two velocities were equal to v and w . However, since the velocity along the OB axis is

$$v + dx \left(\frac{dv}{dx} \right)$$

this increment will transport our element from q to r , through the distance

$$qr = dt \, dx \left(\frac{dv}{dx} \right)$$

parallel to the axis OB . Finally, the increment $dx(dw/dx)$ of the velocity w will transport the element from r to z' through the distance

$$rz' = dt \, dx \left(\frac{dw}{dx} \right)$$

parallel to the third axis OC . From this I conclude that the fluid element which occupied the small linear segment Zz would be transported in the time dt to the segment $Z'z'$, inclined at an infinitely small angle to the OA axis, whose length by virtue of the fact that

$$Z'q = dx \left(1 + dt \left(\frac{du}{dx} \right) \right)$$

would be

$$dx \sqrt{\left(\left(1 + dt \left(\frac{du}{dx} \right) \right)^2 + dt^2 \left(\frac{dv}{dx} \right)^2 + dt^3 \left(\frac{dw}{dx} \right)^2 \right)}$$

Thus, if the terms that contain the square of dt are neglected, the length $Z'z'$ would not differ from $Z'q$ and we would have

$$Z'z' = dx \left(1 + dt \left(\frac{du}{dx} \right) \right)$$

With respect to the inclination of this line to the OA axis, it will suffice to note that it is an infinitely small quantity of the first order and can be expressed as αdt .

14. If the small segment Zz had been taken equal to dy and parallel to the OB axis, by the same reasoning it could have been shown that the fluid which occupied that segment would have been transported to another segment

$$Z'z' = dy \left(1 + dt \left(\frac{dv}{dy} \right) \right)$$

whose inclination to the OB axis was also infinitely small. And if the segment Zz had been taken equal to dz and parallel to the third axis OC , the fluid occupying it would have been transported to another segment

$$Z'z' = dz \left(1 + dz \left(\frac{dw}{dz} \right) \right)$$

which would have been inclined to the OC axis at an infinitely small angle. Thus, if we consider a rectangular parallelepiped $ZPQRzpq'r'$ (Fig. 3) formed by the three sides

$$ZP = dx, \quad ZQ = dy, \quad ZR = dz$$

the fluid occupying that volume would be transported in the time dt to fill a volume $Z'P'Q'R'z'p'q'r'$ differing infinitely slightly from a rectangular parallelepiped whose three sides would be:

$$Z'P' = dx \left(1 + dt \left(\frac{du}{dx} \right) \right), \quad Z'Q' = dy \left(1 + dt \left(\frac{dv}{dy} \right) \right), \quad Z'R' = dz \left(1 + dt \left(\frac{dw}{dz} \right) \right)$$

Since the sides ZP, ZQ, ZR go over into $Z'P', Z'Q', Z'R'$, it is clear that the fluid contained in the first volume will be transported into the other in the time dt .

15. We can now judge whether the volume of fluid occupying the parallelepiped Zz has increased or decreased in the time dt . For this we need only to find the volume or the capacity of each of these two solids. Since the first is a parallelepiped formed by the sides dx, dy, dz , its volume is equal to $dx dy dz$. As for the other, whose plane angles differ infinitely slightly from a right angle, I note that its volume can also be found by multiplying its three sides, since the error due to the infinitesimal distortion of the angles will enter into terms which contain the square of the time element dt and can therefore be neglected. Thus, the volume $Z'z'$ can be represented by the expression:

$$dx dy dz \left(1 + dt \left(\frac{du}{dx} \right) + dt \left(\frac{dv}{dy} \right) + dt \left(\frac{dw}{dz} \right) \right)$$

Anyone still harboring doubts about the reasonableness of this conclusion need only consult my Latin paper "Principia motus fluidorum" in which I calculate this volume without neglecting anything.¹⁶

16. Thus, if the fluid is not compressible, these two volumes should be equal, since the mass occupying the volume Zz would not fit into either a larger or a smaller volume. However, since I propose to examine the problem in the most general possible form and have denoted the density at Z by q , considering q to be a function of the three coordinates and time, I note that to find the density at Z' it will first be necessary to increase the time t by its differential dt ; then, as the point Z' is different from Z , the quantities x, y, z will have to be increased by the small increments $u dt, v dt, w dt$, whence the density at Z' will be:

$$q + dt \left(\frac{dq}{dt} \right) + u dt \left(\frac{dq}{dx} \right) + v dt \left(\frac{dq}{dy} \right) + w dt \left(\frac{dq}{dz} \right)$$

and since the density is inversely proportional to the volume, this quantity will be to q as $dx dy dz$ to

$$dx dy dz \left(1 + dt \left(\frac{du}{dx} \right) + dt \left(\frac{dv}{dy} \right) + dt \left(\frac{dw}{dz} \right) \right)$$

Thus, dividing by dt , we find that consideration of the density leads to the following equation:

$$\left(\frac{dq}{dt} \right) + u \left(\frac{dq}{dx} \right) + v \left(\frac{dq}{dy} \right) + w \left(\frac{dq}{dz} \right) + q \left(\frac{du}{dx} \right) + q \left(\frac{dv}{dy} \right) + q \left(\frac{dw}{dz} \right) = 0$$

17. Here, then, is a very remarkable condition which already establishes a certain relation between the three velocities u, v and w and the fluid density q . Now this equation can be reduced to a simpler form.¹⁷ Thus, $u(dq/dx)$ is no different from (udq/dx) since this form of expression must be taken to mean that in differentiating q only the quantity x is taken to be a variable, and similarly

$$q \left(\frac{du}{dx} \right) = \left(\frac{qu}{dx} \right)$$

from which it follows that

¹⁶ L. Euler, "Principia motus fluidorum," *Novi Commentarii Academiae Imperialis Scientiarum Petropolitanae*, 6 (1756-1757), 271 (1761) = *Opera Omnia*, ser. II. V. 12, pp. 133-168.

¹⁷ In Euler's subsequent exposition the use of round brackets goes beyond the scope of simple partial derivative notation, but the meaning of the operations is still clear; in Euler's notation $d.qu = d(qu)$, etc.

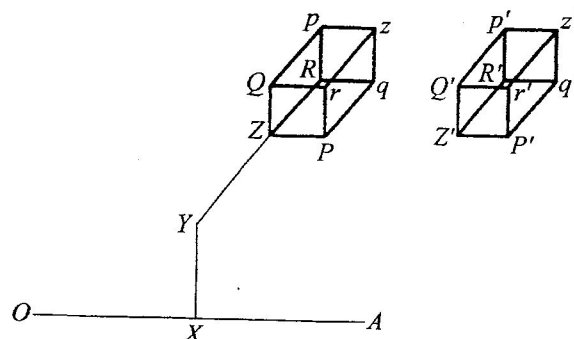


Fig. 3

$$q \left(\frac{du}{dx} \right) + u \left(\frac{dq}{dx} \right) = \left(\frac{u dq + q du}{dx} \right) = \left(\frac{d(qu)}{dx} \right)$$

the differential of the product qu being so understood that only the quantity x is regarded as a variable. Accordingly, the equation obtained can be reduced to the following:

$$\left(\frac{dq}{dt} \right) + \left(\frac{d(qu)}{dx} \right) + \left(\frac{d(qv)}{dy} \right) + \left(\frac{d(qw)}{dz} \right) = 0$$

If the fluid was not compressible, the density q would be the same at both Z and Z' and for this case we would have the equation

$$\left(\frac{du}{dx} \right) + \left(\frac{dv}{dy} \right) + \left(\frac{dw}{dz} \right) = 0$$

which is also that on which I based my Latin paper mentioned above.¹⁸

18. This equation, obtained by considering the continuity of the fluid, already contains a certain relation which must exist between the quantities u, v, w and q . Other relations must be obtained by considering the forces to which each fluid particle is subjected. Thus, in addition to the accelerative forces¹⁹ P, Q, R , which act on the fluid at Z , it is also subjected to the pressure ("pression") exerted from all sides on the fluid element contained at Z . Combining these dual forces, we obtain three accelerative forces in the direction of the three axes. Since the accelerations themselves can be determined by considering the velocities u, v and w , we can derive three equations which, together with that which we have just found, will contain everything that relates to the motion of fluids, so that we shall then have the general and complete laws of the entire science of fluid motion.

19. In order to find the accelerations undergone by a fluid element at Z , we need only compare the velocities u, v, w which currently correspond to the point Z with the velocities corresponding to the point Z' after the lapse of the time dt . Thus, a double change takes place: with respect to the coordinates x, y, z , which receive the increments $u dt, v dt, w dt$, and with respect to time, which increases by dt . Hence it follows that the three velocities at the point Z' are in the direction OA

$$u + dt \left(\frac{du}{dt} \right) + u dt \left(\frac{du}{dx} \right) + v dt \left(\frac{du}{dy} \right) + w dt \left(\frac{du}{dz} \right)$$

in the direction OB

$$v + dt \left(\frac{dv}{dt} \right) + u dt \left(\frac{dv}{dx} \right) + v dt \left(\frac{dv}{dy} \right) + w dt \left(\frac{dv}{dz} \right)$$

in the direction OC

¹⁸ See footnote 16.

¹⁹ Concerning the concept of "accelerative" (body) forces see footnote 5.

$$w + dt \left(\frac{dw}{dt} \right) + u dt \left(\frac{dw}{dx} \right) + v dt \left(\frac{dw}{dy} \right) + w dt \left(\frac{dw}{dz} \right)$$

and hence the accelerations, expressed in terms of the velocity increments divided by the time element dt , will be in the direction OA

$$\left(\frac{du}{dt} \right) + u \left(\frac{du}{dx} \right) + v \left(\frac{du}{dy} \right) + w \left(\frac{du}{dz} \right)$$

in the direction OB

$$\left(\frac{dv}{dt} \right) + u \left(\frac{dv}{dx} \right) + v \left(\frac{dv}{dy} \right) + w \left(\frac{dv}{dz} \right)$$

in the direction OC

$$\left(\frac{dw}{dt} \right) + u \left(\frac{dw}{dx} \right) + v \left(\frac{dw}{dy} \right) + w \left(\frac{dw}{dz} \right)$$

20. We will now seek the accelerative forces acting in these same directions due to the pressure exerted by the fluid on the parallelepiped Zz , whose volume is equal to $dx dy dz$, the mass of the fluid occupying that volume thus being equal to $q dx dy dz$. Since the pressure at the point Z is expressed in terms of the height p , the motive force acting on the face $ZQRp$ is equal to $p dy dz$. For the opposite face $zqrP$ with the area $dy dz$, the height p is increased by its differential $dx (dp/dx)$, obtained on the assumption that only x is variable. Accordingly, this fluid mass Zz is driven in the direction AO by the motive force $dx dy dz (dp/dx)$ or by the accelerative force $1/q(dp/dx)$. Similarly, we find that the fluid mass Zz is subjected to the action of the accelerative force $1/q(dp/dy)$ in the direction BO and to that of the accelerative force $1/q(dp/dz)$ in the direction CO . To these forces we add the given forces P, Q, R . Then the total accelerative forces will be:

in the direction OA

$$P - 1/q(dp/dx)$$

in the direction OB

$$Q - 1/q(dp/dy)$$

in the direction OC

$$R - 1/q(dp/dz)$$

21. Thus, it only remains to equate these accelerative forces with the actual accelerations which we have just found. We then obtain the following three equations²⁰:

$$P - \frac{1}{q} \left(\frac{dp}{dx} \right) = \left(\frac{du}{dt} \right) + u \left(\frac{du}{dx} \right) + v \left(\frac{du}{dy} \right) + w \left(\frac{du}{dz} \right)$$

$$Q - \frac{1}{q} \left(\frac{dp}{dy} \right) = \left(\frac{dv}{dt} \right) + u \left(\frac{dv}{dx} \right) + v \left(\frac{dv}{dy} \right) + w \left(\frac{dv}{dz} \right)$$

$$R - \frac{1}{q} \left(\frac{dp}{dz} \right) = \left(\frac{dw}{dt} \right) + u \left(\frac{dw}{dx} \right) + v \left(\frac{dw}{dy} \right) + w \left(\frac{dw}{dz} \right)$$

If we add to these three equations, first, that obtained from considering the continuity of the fluid, namely

²⁰ Despite the outward resemblance between Euler's equations and modern notation, they have been written here in dimensionless form. As mentioned above, the pressure p is measured as the ratio of the acting pressure to the specific weight $\gamma_0 = \rho_0 g$ of a certain homogeneous auxiliary fluid, the density q is dimensionless ($q = \rho/\rho_0$), the components of the body forces have been divided by the acceleration of gravity g ; the transition from the Eulerian velocities u, v, w to the real velocities U, V, W is effected by means of a transformation of the form $u \rightarrow U/\sqrt{g}$ and the transition from Eulerian time to real time by means of the transformation $t \rightarrow T\sqrt{g}$. (For further details concerning Euler's system of physical units, see G.K. Mikhailov's article in this issue of the Journal).

$$\left(\frac{dq}{dt}\right) + \left(\frac{q \cdot qu}{dx}\right) + \left(\frac{d \cdot qv}{dy}\right) + \left(\frac{d \cdot qw}{dz}\right) = 0$$

and then the equation [of state] which gives the relation between the elasticity [pressure] p , the density q and the other property r which influences the elasticity p , other than the density q , we shall have five equations encompassing the entire Theory of the Motion of Fluids.

22. Whatever the nature of the forces P, Q, R , provided that they are real, it should be noted that $P dx + Q dy + R dz$ is always a total ("réel") differential of a certain finite and determinate quantity,²¹ assuming the three coordinates x, y and z to be variables. Thus, we will always have

$$\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right), \quad \left(\frac{dP}{dz}\right) = \left(\frac{dR}{dx}\right), \quad \left(\frac{dQ}{dz}\right) = \left(\frac{dR}{dy}\right)$$

And if we set this finite quantity equal to S , then, assuming the time t to be constant, we obtain $dS = P dx + Q dy + R dz$ for the case in which the forces P, Q, R also vary with time at the same points. The quantity S expresses what I shall call the effort ("l'effort") of the acting forces²² and is equal to the sum of the integrals of each force multiplied by the elementary interval in the direction of that force or by the small distance through which it would drag a body subjected to its action. This notion of effort is of the utmost importance for the entire theory of both equilibrium and motion, since it makes it possible to see that the sum of all the efforts is always a *maximum* or a *minimum*. This excellent property fits in admirably with the splendid principle of least action whose discovery we owe to our illustrious President, Mr. de Maupertuis.²³

23. The equations just obtained contain four variables x, y, z and t which are absolutely independent of each other since the variability of the first three extends to all elements of the fluid and that of the fourth to all times. Therefore, for the equations to be meaningful, the other variables u, v, w, p and q must be certain functions of the former. For although a differential equation with two variables²⁴ is always soluble ("possible"), we know that a differential equation containing three or more variables is soluble only under certain conditions, by virtue of which a certain relationship must exist between the terms of the equation. Therefore, before we can begin solving the equations, we need to know what sort of functions of x, y, z and t must be used to express the values of u, v, w, p and q in order for these same equations to be soluble.

24. We now multiply the first of the three equations obtained by dx , the second by dy and the third by dz , and since

$$dx \left(\frac{dp}{dx}\right) + dy \left(\frac{dp}{dy}\right) + dz \left(\frac{dp}{dz}\right)$$

represents the differential of p , assuming only time t to be constant, we obtain

$$dS - \frac{dp}{q} = \begin{cases} +dx \left(\frac{du}{dt}\right) + u dx \left(\frac{du}{dx}\right) + v dx \left(\frac{du}{dy}\right) + w dx \left(\frac{du}{dz}\right) \\ +dy \left(\frac{dv}{dt}\right) + u dy \left(\frac{dv}{dx}\right) + v dy \left(\frac{dv}{dy}\right) + w dy \left(\frac{dv}{dz}\right) \\ +dz \left(\frac{dw}{dt}\right) + u dz \left(\frac{dw}{dx}\right) + v dz \left(\frac{dw}{dy}\right) + w dz \left(\frac{dw}{dz}\right) \end{cases}$$

It is now a question of finding the integral of this equation in which time is assumed to be constant. It should be noted that this single equation contains the three equations of which it is composed and that as soon as it is satisfied the conditions of all three will be fulfilled. Thus, if the expression²⁵ $dS - dp/q$ is equal to the three lines, where x, y and

²¹ Euler is thinking here of real body forces possessing a potential (more correctly, a force function). By "finite" quantities (functions) Euler means quantities that do not contain differentials.

²² Euler's "effort" is equivalent to the modern notion of potential.

²³ Maupertuis was president of the Berlin Academy at the time.

²⁴ Here, by variable Euler understands both independent variables and their functions.

²⁵ Here and in what follows, the fraction formed with a slash, introduced only in the 19th century and not used by Euler, is occasionally employed to simplify the typesetting.

z are variables, the portion of $dS - dp/q$ due to the variability of x alone, namely

$$P dx - \frac{dx \left(\frac{dp}{dx} \right)}{q}$$

must necessarily be equal to the first line, and similarly for the other two. The terms

$$\left(\frac{du}{dt} \right), \quad \left(\frac{dv}{dt} \right), \quad \left(\frac{dw}{dt} \right)$$

found by assuming the variability of time t , since they denote certain finite functions, do not prevent time t from now being taken to be constant.

25. Suppose that this equation has already been solved and the quantities u, v, w, q and p have been found as certain finite functions of x, y, z and t . The substitution of these functions in the differential equation, with time t assumed constant, yields an identity. Since after this substitution we will have three types of terms — the first associated with dx , the second with dy and the third with dz , the identity leads us to three equations whence it is clear that although only one differential equation is being considered, it actually has the force of three and determines three of our unknowns. What is also clear is that a differential equation with three variables, such as $L dx + M dy + N dz = 0$, cannot be solved unless a certain relationship exists between the quantities L, M and N . However, since very little work has yet been done on solving these three-variable equations, we cannot hope to obtain a complete solution of our equation until the limits of Analysis have been extended much further.

26. The best approach would therefore be to ponder well on the particular solutions of our differential equation that we are in a position to obtain, as this would enable us to judge which path to follow in order to arrive at a complete solution. I have already pointed out²⁶ that where the density q is assumed to be constant a very elegant solution can be obtained when the velocities u, v and w are such that the differential equation $u dx + v dy + w dz$ can be integrated. Suppose, then, that W is that integral, being a function of x, y, z and time t , and that its differentiation, if t is also taken to be a variable, gives $dW = u dx + v dy + w dz + \Pi dt$. Then the quantities u, v, w and Π will be related as follows²⁷:

$$\left(\frac{du}{dy} \right) = \left(\frac{dv}{dx} \right), \quad \left(\frac{du}{dz} \right) = \left(\frac{dw}{dx} \right), \quad \left(\frac{du}{dt} \right) = \left(\frac{d\Pi}{dx} \right)$$

$$\left(\frac{dv}{dz} \right) = \left(\frac{dw}{dy} \right), \quad \left(\frac{dv}{dt} \right) = \left(\frac{d\Pi}{dy} \right), \quad \left(\frac{dw}{dt} \right) = \left(\frac{d\Pi}{dz} \right)$$

27. Using these equalities, we can reduce our differential equation to the following form:

$$dS - \frac{dp}{q} = \begin{cases} +dx \left(\frac{d\Pi}{dx} \right) + u dx \left(\frac{du}{dx} \right) + v dx \left(\frac{du}{dy} \right) + w dx \left(\frac{du}{dz} \right) \\ +dy \left(\frac{d\Pi}{dy} \right) + u dy \left(\frac{dv}{dx} \right) + v dy \left(\frac{dv}{dy} \right) + w dy \left(\frac{dv}{dz} \right) \\ +dz \left(\frac{d\Pi}{dz} \right) + u dz \left(\frac{dw}{dx} \right) + v dz \left(\frac{dw}{dy} \right) + w dz \left(\frac{dw}{dz} \right) \end{cases}$$

Since here time t is assumed to be constant, using the same hypothesis we will have

$$dx \left(\frac{d\Pi}{dx} \right) + dy \left(\frac{d\Pi}{dy} \right) + dz \left(\frac{d\Pi}{dz} \right) = d\Pi$$

$$dx \left(\frac{du}{dx} \right) + dy \left(\frac{du}{dy} \right) + dz \left(\frac{du}{dz} \right) = du$$

.....

Thus, our equation will become

²⁶ See §§60-67 of the Euler memoir cited in footnote 16.

²⁷ In modern terminology, the function introduced by Euler $W = W(x, y, z, t)$ is the velocity potential; here, the equality of the cross derivatives of W with respect to the coordinates (condition of integrability of dW) is the condition of absence of vorticity.

$$dS - \frac{dp}{q} = d\Pi + u du + v dv + w dw$$

or

$$dp = q(dS - d\Pi - u du - v dv - w dw)$$

Hence, if the density of the fluid is everywhere the same, or $q=g$, as a result of integration we obtain²⁸

$$p = g(C + S - \Pi - 0.5uu - 0.5vv - 0.5ww)$$

28. For brevity, let us set

$$C + S - \Pi - 0.5uu - 0.5vv - 0.5ww = V$$

where it should be noted that the constant C may well contain the time t , since it is considered to be constant in this integration and, as $dp = q dV$, it is clear that the hypothesis $dW = u dx + v dy + w dz + \Pi dt$ also makes our differential equation soluble when the elasticity p depends in some way on the density q only or q is some function of p . It will also be soluble if the fluid is not compressible but the density q varies in such a way that it is an arbitrary function of the quantity V . And in general, if the elasticity p depends both on the density q and on some other quantity represented by the letter r , the hypothesis may also be satisfied provided that r is a function of V . In all these cases, for the motion to exist under this hypothesis it is also necessary for the following condition to be satisfied:

$$\left(\frac{dq}{dt}\right) + \left(\frac{d.qu}{dx}\right) + \left(\frac{d.qv}{dy}\right) + \left(\frac{d.qw}{dz}\right) = 0$$

29. This hypothesis is so general that it seems that there is not a single case that is not included and hence that, generally speaking, the equation $dp = q dV$ — together with the other equations which present hardly any difficulty — incorporates all the foundations of the Theory of the Motion of Fluids. Thus, I concerned myself exclusively with this case in my Latin memoir on the laws of fluid motion²⁹ in which I considered incompressible fluids only and showed that all the cases previously considered, in which the fluid moves through pipes of some description, are contained in this supposition and that the velocities u , v and w are always such that the differential expression $u dx + v dy + w dz$ is integrable. However, I have since noted that there are also cases, even when the fluid is incompressible and everywhere homogeneous, in which this condition does not hold, which is enough to convince me that the solution I have just given is only a particular one.

30. To give an example of a real motion which would be perfectly consistent with all the equations that follow from the laws of Mechanics, but without the expression $u dx + v dy + w dz$ being integrable, let us assume that the fluid is incompressible and everywhere homogeneous, i.e. that q is constant and equal to g , and that there are no forces acting on the fluid, so that $P=0$, $Q=0$ and $R=0$. Then, let $w=0$, $v=Zx$ and $u=-Zy$, where Z denotes an arbitrary function of $(xx + yy)^{1/2}$. It is now obvious that the expression $u dx + v dy + w dz$, which takes the form $-Zy dx + Zx dy$, is integrable only in the case

$$Z = 1/(xx + yy)$$

However, these values [of u , v and w] satisfy all our formulas so that the possibility of this motion cannot be questioned. Since Z is a function of $(xx + yy)^{1/2}$, its differential will have the form $dZ = Lx dx + Ly dy$, where L will again be some function of $(xx + yy)^{1/2}$.

31. Using these values of u , v and w , we obtain

$$\begin{aligned} \left(\frac{du}{dt}\right) &= 0, & \left(\frac{du}{dx}\right) &= -Lxy, & \left(\frac{du}{dy}\right) &= -Z - Lyy, & \left(\frac{du}{dz}\right) &= 0 \\ \left(\frac{dv}{dt}\right) &= 0, & \left(\frac{dv}{dx}\right) &= Z + Lxx, & \left(\frac{dv}{dy}\right) &= +Lxy, & \left(\frac{dv}{dz}\right) &= 0 \\ \left(\frac{dw}{dt}\right) &= 0, & \left(\frac{dw}{dx}\right) &= 0, & \left(\frac{dw}{dy}\right) &= 0, & \left(\frac{dw}{dz}\right) &= 0 \end{aligned}$$

²⁸ The subsequent equations, which generalize the Bernoulli integral, are usually associated with the names of Cauchy and Lagrange.

²⁹ See the memoir cited in footnote 16.

and since $dS=0$, assuming time t to be constant, we have the following differential equation:

$$-\frac{dp}{g} = \left\{ \begin{array}{l} +LZxyy dx - ZZx dx - LZxyy dx \\ -ZZy dy - LZxxy dy + LZxyy dy \end{array} \right\} = -ZZ(x dx + y dy)$$

Consequently,

$$dp = gZZ(x dx + y dy)$$

and, since Z is assumed to be a function of $(xx + yy)^{1/2}$, this equation will definitely be soluble and will yield the integral

$$p = g \int ZZ(x dx + y dy)$$

Of course, the differential equation would also be soluble even if the fluid were subjected to the action of certain arbitrary forces P, Q, R , provided that the expression

$$P dx + Q dy + R dz$$

was a total ("possible") differential equal to dS , since then

$$p = gS + g \int ZZ(x dx + y dy)$$

32. Since these values $u = -Zy, v = Zx$ and $w = 0$ satisfy our differential equation, they should also satisfy the condition contained in the equation³⁰:

$$\left(\frac{dq}{dt} \right) + \left(\frac{d.qu}{dx} \right) + \left(\frac{d.qv}{dy} \right) + \left(\frac{d.qw}{dz} \right) = 0$$

By virtue of the fact that $q=g$, this equation goes over into $-gLxy + gLxy=0$ which, being an identity, satisfies the required conditions. Thus, it is quite possible for a fluid to have a motion such that the velocities of each of its elements are $u = -Zy, v = Zx$ and $w = 0$, although the differential expression $u dx + v dy + w dz$ is not soluble [total differential]; this confirms that there are cases in which fluid motion is possible without this condition, which seemed general, being fulfilled. Thus, the assumption that the differential expression $u dx + v dy + w dz$ is soluble yields only a particular solution of the equations we have found.

33. Clearly, the motion corresponding to this case reduces to a rotational motion about the axis OC and since what has been said about the axis OC can be applied to any other fixed axis, we may conclude that it is possible for a fluid acted upon by any forces whose effort³¹ is equal to S to have a motion about a fixed axis such that the rotational velocities are proportional to some function of the distance to that axis. Thus, if the distance to that axis is denoted by s and the rotational velocity at that distance by ψ ,³² then since $xx + yy = ss$ and $ZZss = \psi\psi$, the pressure there will be expressed by the height

$$p = gS + g \int \frac{\psi\psi ds}{s}$$

Thus, such a motion, which corresponds to that of a vortex ("tourbillon"), is just as possible as those contained in the expression $u dx + v dy + w dz$ when the latter is integrable. No doubt there is an infinity of other motions which satisfy our equations and are also equally possible.

34. Let us now return to our general formulas and, since they are somewhat too complicated, introduce, for greater conciseness, the notation:

³⁰ Strictly speaking, it cannot be said that the values of u, v and w assumed in §30 also satisfy the equation of motion from §31; in reality, this equation determines the corresponding pressure $p=p(s)$ ($s=(xx + yy)^{1/2}$), the continuity equation being satisfied irrespective of the equations of motion.

³¹ See footnote 22.

³² Euler uses the astronomical symbol for the constellation of Taurus to denote this quantity. For technical reasons, this has been replaced by ψ .

$$\left(\frac{du}{dt}\right) + u\left(\frac{du}{dx}\right) + v\left(\frac{du}{dy}\right) + w\left(\frac{du}{dz}\right) = X$$

$$\left(\frac{dv}{dt}\right) + u\left(\frac{dv}{dx}\right) + v\left(\frac{dv}{dy}\right) + w\left(\frac{dv}{dz}\right) = Y$$

$$\left(\frac{dw}{dt}\right) + u\left(\frac{dw}{dx}\right) + v\left(\frac{dw}{dy}\right) + w\left(\frac{dw}{dz}\right) = Z$$

Whatever the nature of the three accelerative forces P , Q and R , by virtue of the fact that³³ $dS = P dx + Q dy + R dz$ the differential equation

$$dp/q = (P - X) dx + (Q - Y) dy + (R - Z) dz$$

in which t is assumed to be constant must be satisfied. Moreover, the continuity of the fluid requires that

$$\left(\frac{dq}{dt}\right) + \left(\frac{d.qu}{dx}\right) + \left(\frac{d.qv}{dy}\right) + \left(\frac{d.qw}{dz}\right) = 0$$

However these two equations are satisfied, there will always be a motion which can actually take place in the fluid.

35. If the fluid is everywhere incompressible and homogeneous, i.e. the density q is constant and equal to g , then, clearly, the differential equation cannot be satisfied unless the differential

$$(P - X) dx + (Q - Y) dy + (R - Z) dz$$

is soluble or total, i.e. unless it can be obtained as a result of the actual differentiation of some finite function of the variables x , y and z , which may also contain the time t , although in the differentiation the latter is assumed to be constant. It is also obvious that this differential expression must be soluble or total when the fluid is compressible and the density q is expressed in terms of some function of the elasticity p . In both cases, if we denote by V the finite quantity whose differential has the form:

$$dV = (P - X) dx + (Q - Y) dy + (R - Z) dz$$

our differential equation will yield either $p/g = V$ or

$$\int \frac{dp}{q} = V$$

In addition, however, for the motion to be possible the other condition derived from the continuity must also be fulfilled.

36. If the fluid is not compressible, but its density q is variable and can be expressed in terms of some function of position, i.e. of the three coordinates x , y , z and time t , it is not sufficient for the expression

$$(P - X) dx + (Q - Y) dy + (R - Z) dz = dV$$

to be integrable; in addition, the integral V must be a function of q . Since $dp/q = dV$ or $dp = qdV$, it is clear that the pressure p cannot have a definite value unless the expression qdV can be integrated. However, it should also be noted that in this case it is not necessary that the expression

$$(P - X) dx + (Q - Y) dy + (R - Z) dz$$

be integrable, only that on being multiplied by a certain function U it becomes integrable. Thus, let

$$U(P - X) dx + U(Q - Y) dy + U(R - Z) dz = dW$$

since $dp/q = dW/U$ or $dp = qdW/U$ for this equation to be soluble it is sufficient that W be a function of q/U or that W be a function of zero dimension of the quantities q and U .³⁴

37. In general, however the elasticity p depends on the density q or on some other property denoted by r which is some function of the coordinates x , y , z that could also contain time t , it is clear from our equation $q = dp/dV$ that the

³³ Here, Euler assumes that all real body forces have a potential $S = S(x, y, z)$.

³⁴ This latter expression is equivalent, in 18th century terminology, to the condition that W should depend only on the ratio q/U .

differential dp must always be divisible by dV , where dV denotes not a total differential but the expression $(P - X) dx + (Q - Y) dy + (R - Z) dz$, this being a consequence of the fact that as a result of division the differentials dx , dy and dz are entirely eliminated from the calculations, because both p and q must always be expressed in terms of finite functions of the variables x , y and z , without their differentials entering into these functions. Now this could not be so unless there were a function U , multiplication by which rendered the expression dV integrable. Let us assume that this integral has the form $\int U dV = W$. Then, clearly, p must be a function of W in order for the expression dp/dV to take a definite value corresponding to the density q .

38. Since $UdV = dW$, we have $q = Udp/dW$. Consequently, if as W we take some function of the coordinates x , y and z , which contains time t among the constants, and if we set p equal to some function of W , namely³⁵

$$p = \varphi(W) \quad \text{and} \quad dp = dW\varphi'(W)$$

we will have $q = U\varphi'(W)$, whence $U = q/\varphi'(W)$. Thus, however the density q is expressed in terms of the elasticity p and some other function r of the coordinates x , y and z , we obtain the value $U = q/\varphi'(W)$ and, consequently, the value $dV = dW\varphi'(W)/q$, which then gives us the following equation:

$$(P - X) dx + (Q - Y) dy + (R - Z) dz = \frac{dW\varphi'(W)}{q} = \frac{dp}{q}$$

This will yield the values of X , Y , Z , from which we must then obtain the values of the velocities u , v and w and when these also satisfy the continuity condition, we shall have a case of possible motion of the fluid.

39. It is to this then that the question of the nature of the expression $(P - X) dx + (Q - Y) dy + (R - Z) dz$ reduces. When the density q is constant or depends only on the elasticity p , this expression must be absolutely integrable and for this it is necessary to determine suitable values of the three velocities u , v and w . However, if the density q is represented by a given function of place and time, the expression must be such that it becomes integrable on multiplication by some given function U . In both cases, then, the velocities u , v and w must be such that the equation

$$(P - X) dx + (Q - Y) dy + (R - Z) dz = 0$$

is soluble. Now we know the conditions under which a differential equation with three variables is soluble and having satisfied these conditions it remains to satisfy that imposed by continuity.

40. These are the conditions by which the functions expressing the three velocities u , v and w must be bounded and the entire study of the motion of fluids reduces to determining, in general form, the nature of those functions which would satisfy the conditions of our differential equation and of continuity. Since the quantities X , Y and Z depend not only on the velocities u , v and w themselves but also on their variability with respect to each of the coordinates x , y and z and, moreover, on time t , this study would appear to be the most far-reaching of those to be encountered in the field of Analysis, and if we are unable to achieve a complete understanding of the motion of fluids, it is not Mechanics or the inadequacy of the known laws of motion but Analysis itself that is to blame, given that the entire Theory of the Motion of Fluids has just been reduced to the solution of analytical equations.

41. Since a general solution must be deemed impossible due to the shortcomings of Analysis, we must content ourselves with the consideration of certain particular cases, especially as the study of several cases seems to be the only means of perfecting our knowledge. Now the simplest case imaginable is, no doubt, that in which the three velocities u , v and w are set equal to zero, that is the case in which the fluid remains at perfect rest and which I dealt with in my previous Memoir. The formulas we have obtained for motion in general also include the case of equilibrium, since when $X=0$, $Y=0$ and $Z=0$ we have

$$\frac{dp}{q} = P dx + Q dy + R dz, \quad \left(\frac{dq}{dt}\right) = 0$$

from which it follows, first of all, that the density q cannot depend on time t , i.e. should remain always the same at the same place. Furthermore, the forces P , Q and R must be such that the differential expression $P dx + Q dy + R dz$ either is integrable, when q is constant or depends only on the elasticity p , or becomes integrable upon being multiplied by some function.

42. In my Memoir on fluid equilibrium I only considered cases of the acting forces P , Q , R for which the differential expression $P dx + Q dy + R dz$ is integrable, since this seemed to be the only case that can occur in Nature. In fact, if the density q is either constant or depends only on the pressure p , the fluid could never be in equilibrium unless this

³⁵ For representing the functional dependence $f(x)$ Euler uses the notation f_x and $f \cdot x$. For the reader's convenience, Euler's notation has everywhere been replaced by the modern $f(x)$.

condition relating to the acting forces were satisfied. However, if it were possible for the acting forces to obey some other law, there could be equilibrium provided that the forces were such that there existed some function U which when multiplied by the expression $P dx + Q dy + R dz$ made that expression integrable, or provided that the differential equation $P dx + Q dy + R dz = 0$ were integrable, for then if the density q is expressed in terms of this function U or in terms of the product of this function U and some arbitrary function of the elasticity p , equilibrium may also exist. However, since these cases may not be possible, I shall not consider them in greater detail.

43. After the case of equilibrium, the simplest state that could exist in a fluid is that in which the entire fluid is in uniform motion in the same direction. Let us see, then, how this state is described by our two formulas. In this case, the three velocities being constant, we set $u=a$, $v=b$ and $w=c$; we also have $X=0$, $Y=0$ and $Z=0$. Then our two equations assume the form:

$$\frac{dp}{q} = P dx + Q dy + R dz$$

and

$$\left(\frac{dq}{dt}\right) + a\left(\frac{dq}{dx}\right) + b\left(\frac{dq}{dy}\right) + c\left(\frac{dq}{dz}\right) = 0$$

Hence it is clear that if the density q is constant, the condition of the second equation is satisfied; however, the first equation cannot be satisfied unless the expression $P dx + Q dy + R dz$ admits integration, just as if the fluid were at rest. Of course, such motion can have no effect on the pressure.

44. If, however, the density q is not constant, let us first see what function of x , y , z and t it must be for the second equation to be satisfied. Here, then, is a very curious analytical question which asks what function of the variables x , y , z and t must be taken for q in order that

$$\left(\frac{dq}{dt}\right) + a\left(\frac{dq}{dx}\right) + b\left(\frac{dq}{dy}\right) + c\left(\frac{dq}{dz}\right) = 0$$

This question would appear to be very difficult to answer if formulated in its broadest possible form. However, since when $a=0$, $b=0$, $c=0$ the quantity q is some function of x , y , z that does not contain time t , if we reduce this case to that of rest by imposing on the volume an equal and opposite motion, then, clearly, after time t the coordinates x , y and z will be transformed by the change into $x - at$, $y - bt$, $z - ct$. From this we conclude that our equation will be satisfied if as q we take some function of the three quantities $x - at$, $y - bt$, $z - ct$. And in fact it is easy to see that such a function satisfies the equation, since

$$dq = L(dx - a dt) + M(dy - b dt) + N(dz - c dt)$$

and, consequently,

$$\left(\frac{dq}{dt}\right) = -aL - bM - cN, \quad \left(\frac{dq}{dx}\right) = L, \quad \left(\frac{dq}{dy}\right) = M, \quad \left(\frac{dq}{dz}\right) = N$$

45. Thus, as I have already noted, in order to satisfy the first equation it is necessary that after multiplication by some function U the differential expression $P dx + Q dy + R dz$ be integrable. Therefore let

$$\int U(P dx + Q dy + R dz) = W$$

where the constant of integration also in some way contains time t . Clearly, the expression $P dx + Q dy + R dz$ will also be integrable if it is multiplied by $Uf(W)$, where U and W are known functions, since the acting forces are assumed to be known. Thus, if q does not depend at all on p , then necessarily $q = Uf(W)$, whence the function of the three quantities $x - at$, $y - bt$ and $z - ct$ must be so determined that it can be reduced to the form $Uf(W)$. If, however, q depends only on p , the expression $P dx + Q dy + R dz$ must be absolutely integrable or $U=1$; then, since p will be found in the form of a function of W , the density q will likewise be a function of W , which must also be a function of the quantities $x - at$, $y - bt$ and $z - ct$, and from this we can deduce the nature of this function.

46. However, it can be seen that, in general, the pressure p must always be a function of W , since otherwise the density q could not be a finite function. Therefore let $p=f(W)$ and $dp=f'(W)$; then, by virtue of the fact that

$$P dx + Q dy + R dz = \frac{dW}{U}$$

we obtain $q = Uf'(W)$. Consequently, this case could not arise unless the density q was proportional to the product of

the quantity U and a function of the pressure p or to the product of the quantity $U\varphi(W)$ and some function of p , where $\varphi(W)$ is used to denote a given function of W . For example, let

$$q = ppU\varphi(W);$$

we then have

$$f'(W) \frac{df(W)}{dW} = f(W)^2 \varphi(W)$$

whence we find that the unknown function $f(W)$ is composed of W , for in this example we have

$$\frac{1}{f(W)} = - \int dW \varphi(W) = \frac{1}{p}$$

and hence p can be expressed in terms of W and, consequently, the quantity q will also be known. If this can be reduced to the form of a function of $x - at$, $y - bt$ and $z - ct$, the assumed state of the fluid will be possible and we shall know the pressure and the density at any time and at any point.

47. An example³⁶ will throw more light on these operations which, as they are not yet sufficiently familiar, might appear overly obscure. Thus, let $P=y$, $Q=-x$ and $R=0$; since

$$\frac{dp}{q} = y dx - x dy$$

we obtain

$$U = \frac{1}{yy}, \quad W = \frac{x}{y} + T$$

where T is some function of time t . Moreover, let $q = pp/yy$; since $dp/pp = (y dx - x dy)/yy$ we obtain

$$\frac{1}{p} = \Theta - \frac{x}{y}, \quad p = \frac{y}{\Theta y - x}$$

where the constant Θ also contains time t . As a result, we have

$$q = \frac{1}{(\Theta y - x)^2}$$

and this expression must be a function of $x - at$ and $y - bt$, since z does not enter into it and this is only possible when $\Theta = a/b$; we then have

$$q = \frac{bb}{(ay - bx)^2}, \quad p = \frac{by}{ay - bx}$$

Thus, neither the pressure nor the density depend on time and at a given point will be always the same. This example shows how the calculations should be performed in other cases that might be imagined.

48. Having dealt with this case in which the three velocities are constant, let us now assume that two velocities v and w vanish, which corresponds to the case in which all the fluid particles move in the direction of the OA axis, so that the trajectory described by each is a straight line³⁷ parallel to the OA axis; this case differs from the previous one in that the velocity u is assumed to vary with respect to both place and time. Since

$$X = \left(\frac{du}{dt} \right) + u \left(\frac{du}{dx} \right), \quad Y=0, \quad Z=0$$

our two equations will take the form:

³⁶ In this example forces not having a potential are considered and the integrating factor U is found for these forces.

³⁷ This is the case of so-called shear flow.

$$\frac{dp}{q} = P dx + Q dy + R dz - dx \left(\frac{du}{dt} \right) - u dx \left(\frac{du}{dx} \right)$$

and

$$\left(\frac{dq}{dt} \right) + \left(\frac{d \cdot qu}{dx} \right) = 0$$

This last equation tells us, first of all, that the expression $q dx - qu dt$ must be integrable, the quantities y and z being assumed to be constant with respect to this integration. Thus, the product of q and $dx - u dt$ must be a total differential, i.e. must be integrable.

49. If the density of the fluid is everywhere and always the same, i.e. if q is a constant equal to g , then, since $(du/dx)=0$, it is clear that the velocity u must be independent of the variable x . Let u be some function of the other two coordinates y, z and time t . Then our differential equation will take the form:

$$\frac{dp}{q} = P dx + Q dy + R dz - dx \left(\frac{du}{dt} \right)$$

where time t is assumed to be constant; thus, this expression must be integrable. Accordingly, if the expression $P dx + Q dy + R dz$ obtained from considering the acting forces is integrable in itself, then $dx(du/dt)$ must also be integrable. The expression du/dt does not contain x , but if it were to contain y and z , the expression $dx(du/dt)$ could not be integrable. Thus, du/dt must not contain y and z . Let Z be some function of y and z , and T some function of time t only; then the quantity $u=Z + T$ will satisfy this condition, whence by virtue of the fact that $P dx + Q dy + R dz = dV$ and

$$du/dt = dT/dt$$

we obtain the following integral:

$$\frac{p}{q} = V - x \left(\frac{dT}{dt} \right) + \text{const}$$

50. In further clarification of this case, it should be noted that each fluid particle Z moves exclusively in the direction ZP parallel to the ZA axis and hence the motion of each fluid element will describe a straight line parallel to that axis, so that for the same element there is no change in the value of the two coordinates y and z . Thus, the motion of each particle will either be uniform or will vary with time in such a way that at each instant all the particles undergo the same changes in their motions, which is obvious from the expression $u=Z + T$. In this case the state of pressure is described by the formula

$$p = gV - gx \left(\frac{dT}{dt} \right) + \text{const}$$

where the constant may contain in some form the time t ; here, the pressure depends not only on the effort³⁸ V but also on the change of velocity undergone by each element of the fluid and, moreover, may vary in some way with time.

51. This case provides me with an opportunity to deal with certain questions which naturally arise and whose clarification is of the utmost importance for the theory of both fluid equilibrium and fluid motion. First of all, surprisingly, a change in the velocity of the fluid can occur without the acting forces P, Q, R helping to produce it. Since such a change could take place even when the acting forces vanish, it is reasonable to inquire how it is produced. Next, it also seems paradoxical that the pressure can vary arbitrarily at any instant, and that irrespective of the aforesaid change to which the motion is subjected. This latter difficulty remains even in the state of equilibrium. Thus, letting the three velocities u, v, w vanish, for incompressible fluids we have the integral

$$\frac{p}{g} = V + \text{const}$$

where the constant may in some way contain the time t .

52. To understand this more clearly, one need only imagine a certain mass enclosed in a vessel. Clearly, the state of pressure depends not only on the acting forces but also on any extraneous forces which might be exerted on the vessel. For, even if there were no acting forces, by means of a piston applied to the fluid one could successively produce every possible state of pressure without the equilibrium being disturbed. This is precisely what we can conclude from

³⁸ See footnote 22.

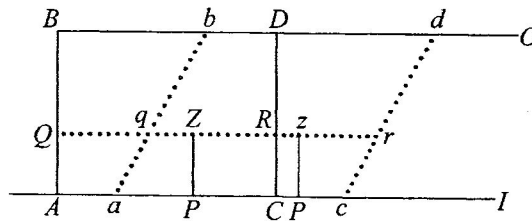


Fig. 4

our formula, which in this case shows that p/g is a function of time t . From this we see that the state of pressure may vary at any instant, irrespective of the equilibrium. However, if for each instant of time the pressure at some point is known, then the pressures at all the other points can be determined, and since the force applied to the piston might now increase and now decrease, the calculations must reflect all these possible changes. The same variability should also be observed when the fluid is subjected to the action of arbitrary accelerative forces, so that at each instant the state of pressure is indeterminate and depends on the force then acting on the piston.

53. Here, then, is a vital difference between the accelerative forces, which act on all the elements of the fluid, and the force of a piston that presses on the fluid. Only the accelerative forces enter into our differential equation, while the piston force enters into the calculations only after integration and only affects the constant of integration. Consequently, in each case the constant must be so determined that at the point at which the piston acts the pressure is exactly equal to the force driving the piston at each instant, and it is for this reason that the constant contains time, so that it can be varied with time at will, as the circumstances require. This variability can always be represented by the action of a piston since, whatever the nature of the case considered, for it to be determined it must always be assumed that at one point at least in the fluid the pressure is known at every instant, and it is precisely this which makes it possible to determine the constant introduced into the calculations as a result of the integration of our differential equation.

54. However, in our case of the motion considered in § 49, let us also assume that the accelerative forces vanish, i.e. that $V=0$, and to make this case perfectly determinate, let us assume that $u=a + \alpha y + \beta t$. Then the equation for the pressure will take the form $p/g = \text{const} - \beta x$. Let us assume, moreover, that this constant is equal to $\gamma + \delta t$, so that $p/g = \gamma + \delta t - \beta x$, and see under what conditions this motion can take place. Since each fluid element moves in the direction of the OA axis, the motion could only take place in a cylindrical pipe laid in the same direction. Let $ABIO$ (Fig. 4) be that pipe and initially, at $t=0$, let the fluid occupy the portion $ABCD$ bounded by cross sections AB and CD perpendicular to the pipe. We will reckon the abscissas x from the point A along the straight line AI and let the pressure p be equal to γg everywhere along the base AB and to $\gamma g - \beta g \cdot AC$ along the other base CD . In the interior of the fluid, however, at some point Z with the coordinates $AP=x$, $PZ=y$, the pressure will be equal to $\gamma g - \beta g x$. Consequently, it is impossible to consider the fluid in the pipe beyond CD , it being necessary to take $AC = \gamma/\beta$ in order that the pressure at CD does not become negative.

55. We will assume that for this determinate fluid mass the length $AC=b$ and the width $AB=CD=c$, the height not entering into consideration since neither the velocities nor the pressures depend on the third coordinate z ; when $\gamma = \beta b$, in the initial state $ABCD$ the pressure is equal to $\beta b g$ on the base AB and zero on the base CD , while at some point Z it is equal to $\beta g(b - x) = \beta g \cdot CP$. We will assume that in this state the fluid has a motion in the direction of the pipe such that the velocity on the line AC is equal to a and that on the line BD equal to $a + \alpha c$, while on some line QR parallel to the direction of the pipe it is equal to $a + \alpha y$, where $AQ=CR=y$. Thus, we imagine that something has caused this motion to be impressed on the fluid and that, at the initial instant, the surface AB is subjected to the force $\beta b g$ exerted by means of a piston, while the other base CD is not subjected to any pressure. However, at subsequent moments of time the forces acting on the end faces could vary arbitrarily. Now this variability is determined by the hypotheses we have just established. Therefore let us see how by virtue of these hypotheses the motion of the fluid will be continued.

56. After the lapse of a time t , all the fluid elements on the line QR will have a velocity in that same direction equal to $a + \alpha y + \beta t$, as a result of which in the time dt they will travel a distance $(a + \alpha y + \beta t)dt$; thus, from the beginning of the motion they will have traveled a distance $at + \alpha yt + 0.5\beta t^2$, and a fluid filament initially at QR will now have advanced to qr , having traversed the distance $Qq = at + \alpha yt + 0.5\beta t^2$. Thus, the filament AC will have arrived at ac , having traveled a distance $Aa = at + 0.5\beta t^2$, while the filament BD will have arrived at bd , having traveled a distance $Bb = at + \alpha ct + 0.5\beta t^2$, so that the fluid mass will now be bounded by the faces ab and cd , which are straight but inclined to the direction of the pipe. The pressure on the face ab at q must now be

$$g(\beta b + \delta t - \beta \cdot Qq) = g(\beta b + \delta t - \beta at - \alpha \beta yt - 0.5\beta \beta tt)$$

and that at r on the face cd

$$g(\beta b + \delta t - \beta \cdot Qr) = g(\delta t - \beta at - \alpha \beta yt - 0.5\beta \beta tt)$$

Thus, we need to visualize pistons which act with these forces on the two end faces ab and cd , and since the pressures are not the same over the entire length of these faces, the pistons must be imagined as being flexible and pliable enough to exert such pressures.

57. This motion would remain the same if in integrating the pressure p we were to take some function of t instead of δt , but then the state of pressure in the fluid mass would be different at each instant of time, even though the assumed motion of the fluid itself would not be affected in any way. Thus, let us assume that $\delta t = \beta at + \alpha \beta ct + 0.5\beta \beta tt$; after a time t the pressure at some point q on the face ab will be $g(\beta b + \alpha \beta(c - y)t)$ and at some point z on the line qr it will be equal to $g(\beta b + \alpha \beta(c - y)t - \beta \cdot qz)$; therefore the pressure at the other end r will be $\alpha \beta g(c - y)t$. Hence, on the face ab the pressure will be equal to $\beta g(b + \alpha ct)$ at a and to βgb at b , while on the other face cd the pressure will be equal to $\alpha \beta gct$ at c and to zero at d . Moreover, each filament QR will move in its own direction with uniform acceleration, i.e. will receive equal increments of velocity in equal times. The study of this particular case could serve to elucidate the calculations to be made in all other cases.

58. Let us now return to the case proposed (§48) and assume the density q to be constant and equal to g , while making the forces P, Q, R such that the fluid could never be in equilibrium. To this end, let $P=0, Q=-x/a$ and $R=-x/a$ and let

$$u = b + \frac{(y + z)t}{a}$$

so that we have

$$\left(\frac{du}{dx}\right) = 0, \quad \frac{dp}{g} = -\frac{x dy + x dz}{a} - \frac{y dx + z dx}{a}$$

whence by integration we obtain

$$\frac{p}{g} = \text{const} - \frac{xy + xz}{a}$$

where the constant may in some way contain time. Thus, it is not possible for the entire fluid mass ever to remain at rest, since even if we set $b=0$ in order to have the fluid at rest at the outset when $t=0$, immediately after that first instant it would be agitated and only the elements for which $y=0$ or $z=0$ or $y + z=0$ would remain at rest; all the others would be set in motion either forward or backward, depending on whether $y + z$ was positive or negative. It is also easy to determine the pressures required to maintain the motion in question.

59. Let, however, the density be no longer constant but variable, i.e. let the fluid be compressible. Then in order for the expression $q dx - qu dt$ to be a total differential we can take for u some function of the variables x, y, z and t . Here, since only the two quantities x and t are regarded as variables and the other two y and z as constants, it will always be possible to assign a quantity s such that $s(dx - u dt)$ is integrable. Let S be that integral; then this condition will be satisfied if we take $q = sf(S)$, where $S = \int s(dx - u dt)$. Furthermore, it is now necessary that the following differential equation be integrable:

$$\frac{dp}{q} = P dx + Q dy + R dz - dx \left(\frac{du}{dt}\right) - u dx \left(\frac{du}{dx}\right)$$

Note that if the forces P, Q, R were to vanish, the pressure p would be a function of x and t and hence the quantity

$$q \left(\left(\frac{du}{dt}\right) + u \left(\frac{du}{dx}\right) \right)$$

would only have to contain the two variables x and t , from which the nature of the function u , although it may also contain y and z , must be determined.

60. Although I have assumed that $v=0$ and $w=0$, these formulas cover all the cases in which all the fluid particles always move in the same direction, the only requirement being that the OA axis be taken in that direction. Therefore we will also be able to solve our equations when the direction of motion is inclined to the three axes, which cannot fail to throw further light on the analysis. To this end, let us consider the true velocity of some fluid particle Z and let that

velocity be equal to ψ^{39} , and since its direction is given with respect to the three axes, certain relations will hold between the velocity components. Let $u = \alpha\psi$, $v = \beta\psi$ and $w = \gamma\psi$; setting $d\psi = Kdt + Ldx + Mdy + Ndz$, we obtain

$$X = \alpha K + \alpha\alpha L + \alpha\beta M + \alpha\psi N$$

$$Y = \beta K + \alpha\beta L + \beta\beta M + \beta\psi N$$

$$Z = \gamma K + \alpha\gamma L + \beta\gamma M + \gamma\psi N$$

Consequently, if, for conciseness, we write $K + \alpha L + \beta M + \gamma N = O$, so that $X = \alpha O$, $Y = \beta O$, $Z = \gamma O$, our equations will take the form:

$$\frac{dp}{q} = P dx + Q dy + R dz - O(\alpha dx + \beta dy + \gamma dz)$$

$$\left(\frac{dq}{dt}\right) + \alpha\left(\frac{d.q\psi}{dx}\right) + \beta\left(\frac{d.q\psi}{dy}\right) + \gamma\left(\frac{d.q\psi}{dz}\right) = 0$$

61. Firstly, let the density $q = g$. As we have seen in § 44, in order to satisfy the equation

$$\alpha\left(\frac{d\psi}{dx}\right) + \beta\left(\frac{d\psi}{dy}\right) + \gamma\left(\frac{d\psi}{dz}\right) = 0$$

the quantity ψ must be some function of the quantities $\alpha y - \beta x$ and $\alpha z - \gamma x$ or $\beta z - \gamma y$ and, in addition, may in some way contain time t . Thus, let ψ be some function of the quantities $\alpha y - \beta x$, $\alpha z - \gamma x$ and t , since the expression $\beta z - \gamma y$ has already been formed from the other two. From this it is easy to see that at each instant the velocity of particles on the same straight line parallel to the direction of motion will be everywhere the same, just as the nature of the hypothesis requires. Hence the differential of ψ will have the following form:

$$d\psi = F dt + G(\alpha dy - \beta dx) + H(\alpha dz - \gamma dx)$$

so that $K = F$, $L = -\beta G - \gamma H$, $M = \alpha G$ and $N = \alpha H$. Consequently, $O = F$ is a function of $\alpha y - \beta x$, $\alpha z - \gamma x$ and t and hence the differential equation, which remains to be solved, will be

$$\frac{dp}{q} = P dx + Q dy + R dz - F(\alpha dx + \beta dy + \gamma dz)$$

62. We will assume that here time t is constant. Then if the expression $P dx + Q dy + R dz = dV$ is integrable in itself, the other part of the equation $F(\alpha dx + \beta dy + \gamma dz)$ must also be integrable, and this could not be so unless F were a function of $\alpha x + \beta y + \gamma z$ and time t . In addition, however, F must also be a function of the quantities $\alpha y - \beta x$, $\alpha z - \gamma x$ and t ; consequently, since the expression $\alpha x + \beta y + \gamma z$ cannot be formed from the expressions $\alpha y - \beta x$ and $\alpha z - \gamma x$, it is clear that the quantity F must be a function of time t only. Consequently, the velocity ψ will have the form $\psi = Z + T$, where Z denotes some function of the two quantities $\alpha y - \beta x$ and $\alpha z - \gamma x$ that does not contain time t , while T is some function of time t only, so that $dT = F dt$. Hence the integral of our differential equation will be

$$\frac{p}{g} = V - F(\alpha x + \beta y + \gamma z) + \text{const}$$

where the constant may in some way contain time t . Together with the relation $\psi = Z + T$, this integral contains everything relating to the motion in the case in question.

63. However, if the density q is not constant, it will be important to obtain the solution of the following equation:

$$\left(\frac{dq}{dt}\right) + \alpha\left(\frac{d.q\psi}{dx}\right) + \beta\left(\frac{d.q\psi}{dy}\right) + \gamma\left(\frac{d.q\psi}{dz}\right) = 0$$

However difficult this may appear, reduction to the previous case shows that the velocity ψ can be some function of the four variables x , y , z and t , while the value of q must be determined as follows. Let us consider, in the general case, the expression

$$s(l dx + m dy + n dz - \psi dt) = dS$$

which becomes integrable after multiplication by s , and let $q = sf(S)$. Then, if we set $df(S) = dsf'(S)$, our expression will

³⁹ Here, as in § 33, Euler's Taurus sign has been replaced by ψ .

take the form:

$$f(S)\left(\frac{ds}{dt}\right) - sf'(S)s\psi + \alpha sf(S)\left(\frac{d\psi}{dx}\right) + \alpha \psi f(S)\left(\frac{ds}{dx}\right) + \alpha \psi sf'(S)ls + \\ \beta sf(S)\left(\frac{d\psi}{dy}\right) + \beta \psi f(S)\left(\frac{ds}{dy}\right) + \beta \psi sf'(S)ms + \gamma sf(S)\left(\frac{d\psi}{dz}\right) + \gamma \psi f(S)\left(\frac{ds}{dz}\right) + \gamma \psi sf'(S)ns$$

and must be equal to zero.

64. First of all, we equate to zero the terms containing $f'(S)$, as a result of which we obtain $1 = \alpha l + \beta m + \gamma n$; after division by $f(S)$ the remaining terms give

$$\left(\frac{ds}{dt}\right) + \alpha \left(\frac{d.s\psi}{dx}\right) + \beta \left(\frac{d.s\psi}{dy}\right) + \gamma \left(\frac{d.s\psi}{dz}\right) = 0$$

which is very similar to the expression proposed; however, it should be noted that the integrability of the quantity ds is subject to the following conditions:

$$\left(\frac{d.s\psi}{dx}\right) = -\left(\frac{d.ls}{dt}\right), \quad \left(\frac{d.s\psi}{dy}\right) = -\left(\frac{d.ms}{dt}\right), \quad \left(\frac{d.s\psi}{dz}\right) = -\left(\frac{d.ns}{dt}\right)$$

whence we obtain

$$\left(\frac{ds}{dt}\right)(1 - \alpha l - \beta m - \gamma n) = 0$$

which is consistent with the previous condition. Thus, provided that $\alpha l + \beta m + \gamma n = 1$, and s is a function that gives $s(l dx + m dy + n dz - \psi dt) = dS$, thereby ensuring integrability, our equation will be satisfied if we take $q = sf(S)$, which means that q/s is some function of S . The quantities l , m and n do not have to be constant, but it will then be necessary to satisfy the condition

$$\alpha \left(\frac{dl}{dt}\right) + \beta \left(\frac{dm}{dt}\right) + \gamma \left(\frac{dn}{dt}\right) = 0$$

which is already contained in the equation $1 = \alpha l + \beta m + \gamma n$.

65. In addition, l , m and n must be functions such that the differential equation $l dx + m dy + n dz - \psi dt = 0$ is soluble, since without this condition it would be impossible to find a multiplier s which made the equation integrable. Thus, if we arbitrarily choose some value for l , the values of m and n will be already determined and we can avoid having to find them. We will set $\alpha l = 1$ or $l = 1/\alpha$; then, necessarily, $\beta m + \gamma n = 0$ and it remains only to find the factor s for which the expression

$$s\left(\frac{dx}{\alpha} - \psi dt\right)$$

is integrable, the two quantities y and z being considered to be constants. Thus, let

$$S = \int s\left(\frac{dx}{\alpha} - \psi dt\right)$$

y and z being contained in S as constants; we can now take $q = sf(S)$, which gives us the same solution as if we had so changed the position of the three axes that one of them coincided with the direction of motion of all the fluid elements. Hence we see that this apparent restriction in no way diminishes the generality of the solution.

66. In the same way it would be possible to study several other particular cases of sometimes greater and sometimes lesser scope, but we would not find a case more general than that in which the three velocities u , v and w are such that the expression $u dx + v dy + w dz$ becomes integrable.⁴⁰ Let S be an integral which also contains time t and let its total differential

⁴⁰ In §§30-33 above, Euler has already pointed out the possibility and given examples of non-potential fluid flows. Truesdell considers that Euler based §66 of his memoir on his previous work (see footnote 16) which was completed before he had discovered the existence of non-potential flows. This seems all the more likely in that, as Truesdell points out, Euler here denotes the velocity potential not by W , as in § 26, but by S , as in his earlier study.

$$dS = u dx + v dy + w dz + \Pi dt$$

Since

$$\left(\frac{du}{dt}\right) = \left(\frac{d\Pi}{dx}\right), \quad \left(\frac{dv}{dt}\right) = \left(\frac{d\Pi}{dy}\right), \quad \left(\frac{dw}{dt}\right) = \left(\frac{d\Pi}{dz}\right)$$

$$\left(\frac{du}{dy}\right) = \left(\frac{dv}{dx}\right), \quad \left(\frac{du}{dz}\right) = \left(\frac{dw}{dx}\right), \quad \left(\frac{dv}{dz}\right) = \left(\frac{dw}{dy}\right)$$

we have

$$X = \left(\frac{d\Pi}{dx}\right) + u \left(\frac{du}{dx}\right) + v \left(\frac{dv}{dx}\right) + w \left(\frac{dw}{dx}\right)$$

$$Y = \left(\frac{d\Pi}{dy}\right) + u \left(\frac{du}{dy}\right) + v \left(\frac{dv}{dy}\right) + w \left(\frac{dw}{dy}\right)$$

$$Z = \left(\frac{d\Pi}{dz}\right) + u \left(\frac{du}{dz}\right) + v \left(\frac{dv}{dz}\right) + w \left(\frac{dw}{dz}\right)$$

and our differential equation now becomes

$$\frac{dp}{q} = P dx + Q dy + R dz - d\Pi - u du - v dv - w dw$$

(the last member of which is absolutely integrable), while the other equation remains as before:

$$\left(\frac{dq}{dt}\right) + \left(\frac{d.qu}{dx}\right) + \left(\frac{d.qv}{dy}\right) + \left(\frac{d.qw}{dz}\right) = 0$$

67. Thus, everything reduces to finding suitable values for the three velocities u , v and w that satisfy our two equations, which contain everything we know about the motion of fluids. For if these three velocities are known, we can determine the trajectory described by each element of the fluid in its motion. Let us consider a particle which at a given instant is located at the point z ; for finding the trajectory which it has already described and which it has yet to describe, since its three velocities u , v and w are assumed to be known, for its position at the next instant we have $dx = u dt$, $dy = v dt$ and $dz = w dt$. Eliminating time t from these three equations, we obtain two more equations in the three coordinates x , y and z which will determine the unknown trajectory of the fluid element now at Z and, in general, we shall know the path which each particle has traveled and has yet to travel.

68. The determination of these trajectories is of the utmost importance and should be used to apply the Theory to each case considered. If the shape of the vessel in which the fluid moves is given, the fluid particles which touch the surface of the vessel must necessarily follow its direction; therefore the velocities u , v and w must be such that the trajectories derived therefrom lie on that same surface.⁴¹ This makes it quite clear how far removed we are from a complete understanding of the motion of fluids and that my exposition is no more than a mere beginning. Nevertheless, everything that the Theory of Fluids contains is embodied in the two equations formulated above (§34), so that it is not the laws of Mechanics that we lack in order to pursue this research but only the Analysis, which has not yet been sufficiently developed for this purpose. It is therefore clearly apparent what discoveries we still need to make in this branch of Science before we can arrive at a more perfect Theory of the Motion of Fluids.

⁴¹ Here, Euler is drawing attention to the fact that in order to calculate the motion of a fluid, in addition to the equations of motion, continuity and state and the initial conditions, we also need the boundary conditions.