Part XIIA. **Contents of the** ***General principles of the motion of fluids*** (E 226, pp. 54—91) (1755).

1 "Having established in my preceding memoir the principles of the equilibrium of fluids in full generality . . ., I propose here to treat the motion of fluids on the same footing . . . It is easy to see that this matter is much more difficult, and that it includes researches which are incomparably deeper. Nevertheless I hope to succeed in so far that if there remain any difficulties, they shall not be on the side of mechanics, but solely on the side of analysis: for this science has not yet been carried to the degree of perfection which would be necessary in order to develop analytic formulae including the principles of the motion of fluids."

2 The properties of fluids to be considered are those mentioned in the preceding memoir, except that if the fluid "is not susceptible of compression, one must distinguish two cases: in the former, all the mass is composed of homogeneous parts whose density everywhere is and remains the same; in the latter, it is composed of heterogeneous parts, and one must know the density of each kind, and the proportion of the mixture...

3 "One must suppose also that the state of the fluid is known at a certain time; and that I shall call the primitive state of the fluid . . ., [in which] one must know the disposition of the particles . . . and the motion which has been impressed upon them . . . But often one knows nothing of a primitive state . . . and then the researches are limited ordinarily to finding the permanent state into which the fluid will come at last, without suffering new changes. But neither this circumstance nor the primitive state change anything in the researches we are going to undertake, and the calculation will remain always the same: it is only in the integrations that one must take them into account in order to determine the constants which each integration brings in.

4 "In the third place, one must count among the given quantities the external forces to whose action the fluid is subject. I call these forces external to distinguish them from the intestine forces with which the particles of the fluid act upon one another, since it is these latter which furnish the principal object of the subsequent researches."

5—8 The ideas [especially that of pressure] which I developed more carefully in the preceding memoir are to be employed here, all quantities being functions of time.

9—15 [Motion, acceleration, and change of volume are discussed as in the *Principles*, except that the element of volume is taken as a parallelepiped and terms $O(dt^2)$ are
16 systematically neglected, as in the earlier work of D'ALEMBERT (see above, p. LIII).] But the density at the point Z' into which the element originally of density q at Z is transported in time dt is

$$q + dt\frac{\partial q}{\partial t} + u\,dt\frac{\partial q}{\partial x} + v\,dt\frac{\partial q}{\partial y} + w\,dt\frac{\partial q}{\partial z}\,, \tag{97}$$

"and thence, since the density is reciprocally proportional to the volume, this quantity will be to q as $dx\,dy\,dz$ to $dx\,dy\,dz\left(1 + dt\frac{\partial u}{\partial x} + dt\frac{\partial v}{\partial y} + dt\frac{\partial w}{\partial z}\right)$," so that

$$(98)\qquad \frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y} + w\frac{\partial q}{\partial z} + q\frac{\partial u}{\partial x} + q\frac{\partial v}{\partial y} + q\frac{\partial w}{\partial z} = 0 \ .$$

Equivalently, 17

$$(99)\qquad \frac{\partial q}{\partial t} + \frac{\partial (qu)}{\partial x} + \frac{\partial (qv)}{\partial y} + \frac{\partial (qw)}{\partial z} = 0 \ .$$

[Accelerations and accelerating forces are calculated as in the *Principles*, except that 18—20
general extraneous forces P, Q, R are employed.] "We have only to equate these accelerat- 21
ing forces to the actual accelerations . . . :

$$(100)\qquad \begin{aligned} P - \frac{1}{q}\frac{\partial p}{\partial x} &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \equiv X \ , \\ Q - \frac{1}{q}\frac{\partial p}{\partial y} &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \equiv Y \ , \\ R - \frac{1}{q}\frac{\partial p}{\partial z} &= \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \equiv Z \ . \end{aligned}$$

If we add to these three equations first the one [(99)] which consideration of the continuity of the fluid has furnished for us, and then that which gives the relation between the elasticity p, the density q, and the other quality r which in addition to the density q influences the elasticity, we shall have five equations which include all the theory of the motion of fluids."

"Real" forces are always derivable from an "effort" [potential] function S, so that 22
$dS = Pdx + Qdy + Rdz$. We may express (100) as an equivalent differential formula 23—24

$$(101)\qquad Pdx + Qdy + Rdz - \frac{dp}{q} = Xdx + Ydy + Zdz \ .$$

"Here then is a differential equation, where the time is taken as constant, and the problem
consists in finding the integral of it. . . But since so far there has been but little work on the 25
solution of such differential equations in three variables, we shall not be able to hope for a complete solution of our equation until the boundaries of analysis have been extended considerably further.

"The best method to follow will be then to weigh well those particular solutions of 26
our differential equation which we are in a position to obtain, for from them we shall be able to judge the route we shall have to take to arrive at a complete solution. But I have already remarked [in the *Principles*] that in the case when the density q is supposed constant one can give a very beautiful solution when the velocities $u, v,$ and w are such
that the differential formula $udx + vdy + wdz$ admits integration. Putting W for this 27
integral, we obtain [as in the *Principles*]

$$(102)\qquad dp = q\left(dS - d\frac{\partial W}{\partial t} - udu - vdv - wdw\right).$$

If $q = \text{const.}$, then

(103) $$p = q\left[C + S - \frac{\partial W}{\partial t} - \tfrac{1}{2}(u^2 + v^2 + w^2)\right].$$

28 More generally, if we put

(104) $$V = C + S - \frac{\partial W}{\partial t} - \tfrac{1}{2}(u^2 + v^2 + w^2) ,$$

where $C = C(t)$, we get from (65)

(105) $$dp = q dV ,$$

whence it is clear that the hypothesis [of existence of a velocity potential] "renders . . . our differential equation possible when the elasticity p depends in any manner on the density q," or vice versa. "It becomes possible also when the fluid is not compressible, but the density q varies in such a way that it is a function of V. And in general, if the elasticity p depends in part on the density q and also on another quantity included in the letter r, this hypothesis can be satisfactory also, provided r be a function of V. But in all these cases, in order that the motion can subsist with this hypothesis," it is necessary that (99) be satisfied also.

29 "This hypothesis is so general that it appears that there would be no case not included in it, and thus that the formula $dp = q dV$, added to the other equations, which present almost no difficulty, includes in general all the foundations of the theory of the movement of fluids. Thus in my Latin memoir on the principles of the motion of fluids, where I considered solely incompressible fluids, I adopted this case solely, and I showed that all cases treated up to the present, in which the fluid moves in arbitrary tubes, are included in this assumption, and that the velocities u, v, and w are always such that the differential formula $udx + vdx + wdz$ becomes integrable. Nevertheless I have since noticed that there are also cases, even when the fluid is incompressible and homogeneous throughout, where this condition does not hold at all. That is enough to convince us that the solution which I have just given is only a particular one.

30 "To give an example of a real motion which is perfectly in accord with all the formulæ which the principles of mechanics have furnished, without nevertheless the formula $udx + vdy + wdz$ being integrable, let the fluid be incompressible . . . and $P = 0$, $Q = 0$, and $R = 0$. Next let

(106) $$w = 0 , \qquad v = Zx , \qquad u = -Zy ,$$

where Z stands for an arbitrary function of $\sqrt{x^2 + y^2}$, and it is evident that the formula $udx + vdy + wdz$, which becomes $-Zydx + Zxdy$, is integrable only in the case

(107) $$Z = \frac{1}{x^2 + y^2} .$$

Nevertheless these values satisfy all our formulae, so that one could not cast in doubt the
31 possibility of such a motion." From (107) and (101) we get

(108) $$-\frac{dp}{q} = -Z^2(xdx + ydy) .$$

"Since Z is supposed a function of $\sqrt{x^2 + y^2}$, this equation without a doubt will be possible and will give as its integral $p = q \int Z^2(x dx + y dy)$. One sees that the differential equation would be possible even if the fluid were subjected to arbitrary forces P, Q, R, provided that $P dx + Q dy + R dz$ were a possible differential $= dS$, for then one would have

$$p = qS + q \int Z^2(x dx + y dy) \text{ .''} \tag{109}$$

The condition (99) is satisfied also. "Thus the supposition of the possibility of the differential 32
formula $u dx + v dy + w dz$ furnishes only a particular solution of the formulæ which
we have found." The motions just considered are vortices about the z-axis, in which, if 33
the speed of rotation at distance s from the axis is ȣ, we have $p = qs + q \int ȣ^2 ds/s$.

In general, either for a homogeneous incompressible fluid or for one in which $q = q(p)$, 34—35
it is evident that the differential equation (101) cannot hold unless there is a function V, possibly depending on t, such that

$$dV = (P - X)dx + (Q - Y)dy + (R - Z)dz \text{ .} \tag{110}$$

Then our differential equation will furnish

$$\int \frac{dp}{q} = V \text{ .} \tag{111}$$

For an incompressible but inhomogeneous fluid, it is necessary that there exist functions 36
U and W such that

$$U[(P - X)dx + (Q - Y)dy + (R - Z)dz] = dW \text{ ,} \tag{112}$$

whence (101) becomes $dp = q dW/U$, for the possibility of which it suffices that
$W = W(q/U)$. In fact, in full generality there must be a function U such that (112) 37
holds, for otherwise the density q cannot have a determinate value. If we take W as an 38
arbitrary function and set $p = \varphi(W)$, then $U = q/\varphi'(W)$, and hence

$$(P - X)dx + (Q - Y)dy + (R - Z)dz = \frac{dW \varphi'(W)}{q} \text{ ,} \tag{113}$$

"whence one will obtain the values X, Y, Z, from which at last we shall have to seek the value of the velocities u, v, and w; and when these satisfy also the condition of continuity, one will have a case of a possible motion of a fluid." In all cases the equation

$$(P - X)dx + (Q - Y)dy + (R - Z)dz = 0 \tag{114}$$

must become possible, and also (99) must be satisfied. "These are the conditions by which 40
the functions which express the three velocities u, v, and w must be limited, and all research on the motion of fluids reduces to determining the general nature of these functions . . . This research appears to be the deepest which is to be found in analysis. And if it is not permitted to us to penetrate to a complete knowledge concerning the motion of fluids, it is not to mechanics, or to the insufficiency of the known principles of motion,

which we must attribute the cause. It is analysis itself which abandons us here, since all the theory of the motion of fluids has just been reduced to the solution of analytic formulæ.

41 "Since a general solution must be judged impossible from want of analysis, we must be content with the knowledge of some special cases, and that all the more, since the development of various [special] cases seems to be the only way of bringing us at last to
42 a more perfect knowledge." A state of rest yields a particular solution. In my preceding memoir I considered only the case of forces such that $Pdx + Qdy + Rdz$ is complete, "since this case would appear to be the only one which can take place in nature." But equilibrium would be possible also if $q = f(U)\,g(p)$, where U is an integrating factor for $Pdx + Qdy + Rdz$. "But since such cases are perhaps not possible, I do not pause to develop them more fully."

43 The case of uniform motion is also included. But for compressible fluids, we are led to "a very curious analytic question." *viz.*, to find a function q such that

(115) $$\frac{\partial q}{\partial t} + a\frac{\partial q}{\partial x} + b\frac{\partial q}{\partial y} + c\frac{\partial q}{\partial z} = 0 ,$$

where a, b, and c are constants. Guided by the case of rest, we expect that

$$q = q(x - at,\ y - bt,\ z - ct) ,$$

45—47 and we can easily assure ourselves that such a function is indeed a solution. In this case, one can determine p also, as for example in the case when $P = y$, $Q = -x$.

48 In the case $v = w = 0$, (99) reduces to the condition that $dx - udt$ shall be
49 exact. In the case $q =$ const., if (110) holds, it must follow that

(116) $$u = Z(y, z) + T(t) , \quad \frac{p}{q} = V - x\frac{dT}{dt} + C(t) .$$

50—53 The particles move along parallel straight lines. In this example we can see how it is possible that the velocity and pressure can change even if there are no forces P, Q, R, since the forces exerted by boundaries, such as a piston, enter the calculation only after
54—57 the integrations have been effected. Consider the special case $u = a + \alpha y - \beta t$, $\dfrac{p}{q} = \gamma + \delta t - \beta x$, where $a, \alpha, \beta, \gamma, \delta$ are constants. These constants can then be interpreted in terms of the pressures on planes $x =$ const., and the motion may be regarded as produced by the action of appropriate flexible and movable pistons.

58 A simple example shows that solutions of the equations of motion can exist when the forces are such that equilibrium is impossible.

59 In the case of variable density, notice that one can take u as a perfectly arbitrary function of x, y, z, and t, since there always exists a function s such that $s(dx - udt)$ is integrable, $= dS$, say. Then taking $q = sf(S)$ satisfies the condition of § 48. The condition (101), in the case when $P = Q = R = 0$, then requires that p and
60—65 $q\left(\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x}\right)$ shall be functions of x and t only. While it is true that the direction of the

x-axis is arbitrary, we can also verify directly that the foregoing analysis remains valid for the case $u = \alpha ♉$, $v = \beta ♉$, $w = \gamma ♉$, where α, β, γ are constants.

"In the same fashion one could develop several other particular cases, some of 66 greater, some of lesser extent; but one will not find any which are more general than the one in which the formula $udx + vdy + wdz$ becomes integrable."

Having the three velocities u, v, w which satisfy our equations [(102) and (99)], 67 in order to determine the paths which the particles have travelled and are still to travel, one must eliminate the time from

$$dx = udt\,, \qquad dy = vdt\,, \qquad dz = wdt\,. \tag{117}$$

"The determination of these paths is of the highest importance and must serve to apply 68 the theory to each particular case proposed. For the shape of the vessel in which the fluid moves is given; the particles which touch the surface of the vessel must necessarily follow its direction; and thus the velocities u, v, w must be such that the paths derived from them fall upon the surface itself. But here we see well enough how far distant we yet are from the complete knowledge of the motion of fluids, and that which I have just explained contains only a feeble beginning. Nevertheless, all that the theory of fluids includes is contained in the two equations presented above [(101) and (99)], so that it is not the principles of mechanics which we lack in the pursuit of these researches, but solely analysis, which is not yet sufficiently cultivated for this purpose. And thus we see clearly what discoveries remain for us to make in this science before we can arrive at a more perfect theory of the motion of fluids."

Part XIIB. **Comments on the paper summarized above.**

In large part this paper consists in shorter and more lucid derivation and discussion of generalizations of some of the results in the *Principles*, to which it several times refers. The description of R. DUGAS[1]), "mémoire si parfait qu'il n'a pas vieilli d'une ligne," is almost literally correct. The main departures from the *Principles* are, first, that all the work is carried out in three dimensions from the start, and, second, that the fluid is not generally supposed incompressible. While EULER still measures pressure in units of length, he has abandoned his earlier choice of the units of velocity: from this point onward his velocities are to have the accustomed modern meaning. Thus, for example, the factor $\frac{1}{2}$ now appears in the "BERNOULLI equation" (103) (*cf.*, *e. g.*, (63)).

EULER's definitive formulation of fluid mechanics consists in the differential equations (99) and (100), along with the thermal equation of state $p = p(q, r)$. Thus the six unknowns u, v, w, p, q, r are subject to only five conditions. When EULER says that they "include all the theory of the motion of fluids" he is nevertheless right, since the remaining condition, the energy equation, is extra-mechanical; it was not to be derived in the appropriate generality for more than a century. The degenerate "barotropic" case when r does not appear at all was to be almost the sole object of hydrodynamical re-

1) *Histoire de la mécanique*, Neuchâtel and Paris (1950), see Ch. VIII, § 6.

searches for 150 years; in this case, EULER's system becomes determinate. In treating compressible fluids, however, EULER usually supposed r to be a given function of position, the "degree of heat" being maintained by appropriate sources.

The emphatic cries for new researches in *pure analysis* which bracket and divide the memoir (§§ 1, 40 and 68) are perhaps directed toward those who encouraged FREDERICK II in his contempt for pure mathematics and fear of the infinitesimal calculus[1]). It is neither the first nor the last of such fruitless defenses. Those for whom they are intended are unlikely to notice, unable to comprehend them. Not content with having brought to his court the greatest genius in Europe, for a salary far less than he paid to the overblown MAUPERTUIS or offered to the philosophical D'ALEMBERT, the literary king expected EULER to supervise the laying of aqueducts. Unfortunately EULER was willing and able to undertake such tasks, thus giving FREDERICK occasion for the complaint that the work was not well done.

In this paper EULER has begun to get a grasp of what can be expected of the solution of a partial differential equation. His remarks on the "constants" (§§ 3, 53, *et passim*) are among the earliest to indicate the role of the initial or boundary conditions in determining the appropriate integral. In demonstrating the invariance of solutions representing uniform motion (§§ 60—65) EULER does not merely transform all the former results to a new co-ordinate system, but instead effects the entire integration anew, directly by means of ingenious devices. The linear equation (115) is doubtless the first of its type to appear. EULER's solution is motivated by the hydrodynamical problem; LAGRANGE's well known method of characteristics rests upon a straightforward extension of EULER's idea[2]). The equations of the characteristics, or, in kinematical terms, the paths of the particles, are EULER's eq. (117).

The argument leading to (97) is equally applicable to any flow quantity F, yielding what is now called the "material derivative," usually designated by STOKES's notation $\frac{DF}{Dt}$.

The emphasis on inverse methods of solution (§§ 26, 41) echoes § 66 of the *Principles*. The particular solutions obtained, however, are almost entirely different. The velocity potential W is mentioned, but the attempt to consider solutions of $\nabla^2 W = 0$ has been abandoned entirely. Instead, the emphasis is on motions in which a velocity potential does *not* exist, correcting the error of D'ALEMBERT which EULER had repeated in the *Principles*. Indeed, the main original contribution made by this paper is its demonstration that existence of a velocity-potential is quite a special circumstance. The proof is achieved by exhibiting the counter examples of the simple vortex flows (§§ 30—33) and the motions which are now called "generalized POISEUILLE flows" (§§ 48—58). In this paper the latter occur for the first time altogether; the former, for the first time in such generality. The reaffirmation in § 66 of the existence of a velocity-potential, in direct contradiction to the statements and the analysis just mentioned, is doubtless a relic of an earlier version of the paper, allowed to stand by oversight. This suspicion is supported by the use of S

1) V. pp. 90, 172—173, 175—176 of OTTO SPIESS, *Leonhard Euler*, Frauenfeld and Leipzig (1929).

2) §§ 10—11 of "*Mémoire sur la théorie du mouvement des fluides*," Nouv. mém. acad. sci. Berlin **1781**, 151—198 (1783) = Oeuvres 4, 695—748.

for the velocity potential in this passage, as in the *Principles*, instead of W as in the earlier parts of this paper.

The inverse method projected in §§ 37—39 leads to a heroic attempt at general solution. The result which EULER's reasoning implies, but which he does not actually state, is the following: given an acceleration field $\boldsymbol{a}$, a necessary but not sufficient condition that the velocity field which gives rise to it be a possible flow of an ideal fluid subject to extraneous force $\boldsymbol{f}$ per unit mass is

$$(\boldsymbol{f} - \boldsymbol{a}) \cdot \operatorname{curl}(\boldsymbol{f} - \boldsymbol{a}) = 0\,. \tag{117a}$$

The special case $\operatorname{curl} \boldsymbol{a} = 0$ leads to (87), the usual vorticity equation. It is EULER's insistence on full generality which prevents him from deriving again his former result (61). But in § 25 he appears to know less about what are now called "PFAFFIAN forms" than he did at the time of writing the earlier paper.

EULER's opinion on whether or not the extraneous forces occuring in nature are derivable from a potential varies from one section to another. The example in § 47 showing that equilibrium can be possible under non-conservative forces is rather artificial: the lines of force are concentric circles, but nevertheless the density and pressure are so adjusted that rest or uniform motion results. In the example treated in § 58 it is not actually proved that equilibrium is impossible, since only a special class of flows is considered.

In § 22 EULER again compliments MAUPERTUIS, but the phrase "de la dernière importance dans toute la Théorie" is an empty flourish, since EULER nowhere in any of his papers on fluid mechanics makes the slightest use of the principle of least action or any related idea.

The statements regarding $dx - udt$ in §§ 48 and 49 are equivalent to the following: in any unidirectional flow there exists a function $\psi(x, y, z, t)$ such that

$$q = -\frac{\partial \psi}{\partial x}\,, \quad qu = \frac{\partial \psi}{\partial t}\,. \tag{118}$$

This fact was later exploited by W. KIRCHHOFF[1]).

Part XIIIA.* Contents of the *Sequel to the researches on the motions of fluids (E 227, pp. 92—132) (1755).

"Since in my two preceding memoirs I reduced all the theory of fluids . . . to two 1
analytic equations, the consideration of these formulæ appears to be of the greatest importance, for they include not only all that has been discovered by methods very different and for the most part slightly convincing . . . but also all that one could desire

1) "*Reduktion simultaner partieller Differentialgleichungen bei hydrodynamischen Problemen,*" J. reine angew. Math. **164**, 183—195 (1930).