Berlin Academy on 31 August 1752, and on this ground ENESTRÖM assigned the work published under the title "Principia motus fluidorum" to the same year.

We may conjecture that when in the paper we have just considered EULER after considerable manipulation reached the equation (37), he must have realized that a more direct derivation in terms of the spatial co-ordinates x and y would be easy: he had only to replace the intrinsic co-ordinates in the *Gunnery* (above, p. XL) by a fixed system. He may also have realized the connection with D'ALEMBERT's equation (25), but this is less likely, since the correct and important results of D'ALEMBERT could hardly be distinguished through his dense mist of calculation, philosophy, and error. In any case, it is likely that EULER then threw aside the manuscript on rivers and started afresh on the new plan. The resulting memoir is so important in the history of rational mechanics that we shall now analyse it in detail, even though in some parts the work is very close both to that in the earlier paper which was published later and to that which generalizes it in later papers, published earlier.

Part XA. Contents of ***Principles of the motion of fluids*** (E 258, pp. 133—168) (1752).

Summary

"Here are treated the elements of the theory of the motion of fluids in general, where the whole matter is reduced to this: given a mass of fluid, either free or confined in vessels, when an arbitrary motion shall have been impressed upon it, and meanwhile it is acted upon by arbitrary forces, the motion in which its several particles are to travel shall be determined, and at the same time the pressure with which the several parts act, as well mutually upon each other as also upon the sides of a vessel, shall be ascertained." The paper is divided into two parts. In the former, the case when the fluid breaks up into drops being first excluded, "the motion must be restricted by this rule, that the several ultimate portions must retain ever the same volume; and by this principle the general expressions of motion for the several elements of the fluid are restricted . . . In the second part the author proceeds to the determination of the motion of a fluid produced by arbitrary forces, in which matter the whole investigation reduces to this, that the pressure with which the parts of the fluid at the several points act upon one another shall be ascertained, which pressure is indicated most conveniently, as is customary for water, by a certain height; which is to be understood thus, that the separate elements of fluid sustain a pressure the same as if they were pressed by a heavy column of the same fluid, whose height is equal to that amount." This pressure varies from one point to another. From it, together with the given forces acting on the whole mass, "the acceleration of the several elements, or their retardation, can be assigned [for] the motion, all which determinations are expressed by the author through differential formulae. But indeed most frequently the full development of these formulae is involved in the greatest difficulties. But nevertheless this whole theory has been reduced to pure analysis, and what remains to be completed in it depends solely upon subsequent progress in analysis. Thus it is far from true that purely analytic researches are of no use in applied mathematics; rather, important additions to pure analysis are now required."

First part. Fluids are extremely different from solids, because the motion of a few 1
particles by no means determines that of the rest. In flexible solids there are at least 2
certain laws governing the bending. But in fluid bodies, "whose particles are united among 3
themselves by no bond, the motion of the various particles is much less restricted . . .
Nevertheless it cannot happen that the motion of all the particles of a fluid is governed 4
by no laws whatever," or that any arbitrary motion is possible. The particles are impenetrable,
and thus "an infinite number of motions must be excluded." Those motions which are 5
not impossible I shall call possible. "We must then find what characteristic is appropriate
to possible motions, separating them from impossible ones. When this is done, we shall
have to determine which one of all possible motions in a certain case ought actually to
occur. Plainly we must then turn to the forces which act upon the water, so that the
motion appropriate to them may be determined from the principles of mechanics."

To find this characteristic [of possible motions] "I shall assume the fluid to be such 6
as never to permit itself to be forced into a lesser space, nor can its continuity be inter-
rupted. Once the theory of fluids has been adjusted to fluids of this nature, it will not be
very difficult to extend it also to those fluids whose density is variable and which do not
necessarily require continuity." Incompressibility is to be required of each portion of 7
the fluid. When this condition is satisfied, the quantity of space occupied by a fluid 8
element does not change during motion through "the least little time."

First let the motion of the several points take place in a plane, and let their velocities 9—10
resolved along two rectangular directions x, y be u, v, so that the true speed is
$\sqrt{u^2 + v^2}$, while the angle at which the direction of the velocity is inclined to the x-axis
is Arctan $\frac{v}{u}$. We shall wish to use $\partial u/\partial x$, $\partial u/\partial y$, $\partial^2 u/\partial x^2, \ldots,$ but we must take care 11—12
not to regard them as ratios of exact differentials; these notations, first introduced by
FONTAINE, facilitate calculation. A particle infinitely near to that whose twin velocities 13
are u, v will have velocities $u + (\partial u/\partial x)\, dx + (\partial u/\partial y)\, dy$, $v + (\partial v/\partial x)\, dx + (\partial v/\partial y) dy$,
and in an infinitely short time dt it moves through distances

$$dt\left[u + \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy\right], \quad dt\left[v + \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy\right].$$

"Having noted these things, let us consider a triangular element lmn of water, and 14
let us seek the location into which it . . . is carried by the motion in the time dt." Two of
the sides are taken parallel to the directions x, y. Since no particular relation between
dx and dy is assumed, "it is plain . . . that in thought the whole mass of fluid may be
divided into elements of this sort, so that what we determine for one in general thus
extends equally to all." The distances travelled by the vertices are calculated as in § 13. 15
Since the triangle is infinitely small, "its sides cannot receive any curvature from the 16
motion," and it is carried into a new triangle pqr. The motion must be such that the
area of the triangle pqr is equal to the area of the triangle lmn. The area of pqr is 17—19
expressed by means of the areas of three trapezoids. Equating the two areas yields

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial(u, v)}{\partial(x, y)} dt = 0 . \tag{41}$$

20 Since the term in dt in (41) vanishes when the coefficient is finite, we obtain

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\,. \tag{42}$$

"In this consists the criterion of possible motions . . ., and unless this condition holds, the motion of the fluid cannot take place."

21—25 In the general case, when the motion is not confined to a single plane, completely analogous statements are to be made. The components of velocity along three perpendicular directions x, y, z are u, v, w, and the true speed will be called $V=\sqrt{u^2+v^2+w^2}$.
26 Let us consider the motion of a right triangular pyramid $\lambda\mu\nu o$, three of whose sides are
27—35 normal to the directions of x, y, z. The calculation of the volume of the pyramid $\pi\varphi\varrho\delta$ into which it is carried in the time dt is accomplished by dividing it into four prisms, the base area of each of which is obtained through subdivision into three trapezoids. Equating the two volumes yields

$$0=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}+dt\left[\frac{\partial(u,v)}{\partial(x,y)}+\frac{\partial(v,w)}{\partial(y,z)}+\frac{\partial(u,w)}{\partial(x,z)}\right]+dt^2\,\frac{\partial(u,v,w)}{\partial(x,y,z)}\,. \tag{43}$$

36 "Rejecting the infinitely small terms, we get this equation:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0\,. \quad \ldots \tag{44}$$

It is obtained simply by this condition, that in the motion no part of the fluid is carried into a greater or lesser space, but perpetually the continuity of the fluid as well as the
37 same density is conserved." This condition must be satisfied at every instant, since
38 "up to now I have considered the time simply as a constant quantity." If we wish to regard time as variable, we obtain the same condition to be satisfied at any given instant.

39 *Second part.* Out of all possible motions, the working of forces produces the actual
40 motion. Considering again first the case when the motion occurs in a single plane, we must take account of changes in time:

$$\begin{aligned} du&=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial t}dt\,,\\ dv&=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial t}dt\,. \end{aligned} \tag{45}$$

41 "Since therefore during the passage of time dt the point . . . travels a distance $u\,dt$ parallel to the x-axis, a distance $v\,dt$ parallel to the y-axis, in order to obtain the increments in the velocities u and v of the point . . . for dx and dy we must write the distance $u\,dt$ and $v\,dt$, whence will arise these true increments of the velocities:

$$\begin{aligned} du&=\frac{\partial u}{\partial x}u\,dt+\frac{\partial u}{\partial y}v\,dt+\frac{\partial u}{\partial t}dt\left[=a_x dt\right],\\ dv&=\frac{\partial v}{\partial x}u\,dt+\frac{\partial v}{\partial y}v\,dt+\frac{\partial v}{\partial t}dt\left[=a_y dt\right]. \end{aligned} \tag{46}$$

Hence the accelerating forces which have the power to produce these accelerations will be:

$$(47)\quad \begin{aligned} &\text{Accel. force parallel to } x\text{-axis} = 2\left(\frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v + \frac{\partial u}{\partial t}\right)\left[= 2a_x\right],\\ &\text{Accel. force parallel to } y\text{-axis} = 2\left(\frac{\partial v}{\partial x}u + \frac{\partial v}{\partial y}v + \frac{\partial v}{\partial t}\right)\left[= 2a_y\right], \end{aligned}$$

to which therefore the forces in fact acting upon the particle of water ought to be equal.

"Among the forces, moreover, which in fact work upon the particles of water, the 42 first to be considered comes gravity; whose effect, however, if the plane of motion is horizontal, is to be taken as nothing. But if however the plane is inclined, . . . by gravity arises a constant accelerating force, of magnitude, say α. Then we must not neglect friction, which often hinders the motion of water not a little. Although its laws have not yet been explored sufficiently, nevertheless, following the friction of solid bodies, probably we shall not wander too far astray if we set the friction everywhere proportional to the pressure with which the particles of water press upon one another.

"First, however, must be brought into the calculation the pressure with which the 43 particles of water everywhere mutually act upon each other, by means of which every particle is pressed together on all sides by its neighbors; and in so far as this pressure is not everywhere equal, to that extent motion is communicated to that particle. The water simply will be put everywhere into a certain state of compression similar to that in which quiet water at a certain height finds itself. Therefore this height, at which in quiet water the water is found to be in a like state to compression, will most conveniently be employed for representing the pressure at an arbitrary point l of the fluid. Therefore let that height, or depth, expressing the state of compression at l, be p, a certain function of the co-ordinates x and y, and if the pressure at l varies also with the time, the time also will enter into the function p.

"Let us consider a rectangular element of water, $lmno$, whose sides are $lm = no = dx$ 44 and $ln = mo = dy$, whose area $= dxdy$. Now when the pressure at $l = p$, the pressure at $m = p + \frac{\partial p}{\partial x}dx$, at $n = p + \frac{\partial p}{\partial y}dy$, and at $o = p + \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$. Thus the side lm is pressed by a force $= dx\left(p + \frac{1}{2}\frac{\partial p}{\partial x}dx\right)$, while the opposite side no will be pressed by a force $= dx\left(p + \frac{1}{2}\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy\right)$; therefore by these two forces the element $lmno$ will be impelled in the direction ln by a force $= -\frac{\partial p}{\partial y}dxdy$. Moreover, in a similar manner from the forces $dy\left(p + \frac{1}{2}\frac{\partial p}{\partial y}dy\right)$ and $dy\left(p + \frac{\partial p}{\partial x}dx + \frac{1}{2}\frac{\partial p}{\partial y}dy\right)$ which act on the sides ln and mo will result a force $= -\frac{\partial p}{\partial x}dx\,dy$ impelling the element in the direction lm." Thus calculating the accelerating forces [and equating them to 45 (47)], we shall have these equations:

$$\alpha - \frac{\partial p}{\partial x} = 2\left(\frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v + \frac{\partial u}{\partial t}\right)\left[= 2a_x\right],$$

(48)

$$-\frac{\partial p}{\partial y} = 2\left(\frac{\partial v}{\partial x}u + \frac{\partial v}{\partial y}v + \frac{\partial v}{\partial t}\right)\left[= 2a_y\right],$$

which we collect in the form

(49) $$dp = \alpha dx - 2a_x dx - 2a_y dy + \frac{\partial p}{\partial t}dt ,$$

"which differential must be complete, or integrable."

46—47 "From the nature of complete differentials" it is thus necessary that (24) holds, whence follows [by rearrangement]

(50) $$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = 0 ,$$

"which $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$ plainly satisfies: so that $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$. Since therefore this condition requires that $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$, it appears finally that this differential formula, $udx + vdy$,

48 must be complete, in which therefore the criterion of actual motions consists. . . . This criterion is independent from the preceding, . . . and therefore even if the fluid in motion changes its density, as happens in the motion of elastic fluids such as air, this property will hold nonetheless, *viz.*, that $udx + vdy$ shall be a complete differential. . .

49 "Hence now we shall be able to ascertain the pressure p itself, which is absolutely necessary for perfectly determining the motion of the fluid. Since we have found that $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$, [from (46) and (49) follows]

(51) $$dp = \alpha dx - 2udu - 2vdv + 2\frac{\partial u}{\partial t}udt + 2\frac{\partial v}{\partial t}vdt - 2\frac{\partial u}{\partial t}dx - 2\frac{\partial v}{\partial t}dy + \frac{\partial p}{\partial t}dt \ldots$$

But therefore if we wish to ascertain for the present time the pressure at the several points of the fluid, taking no account of its variation arising with the time, we shall have to consider this equation:

(52) $$dp = \alpha dx - 2\,udu - 2\,vdv - 2\frac{\partial u}{\partial t}dx - 2\frac{\partial v}{\partial t}dy \text{ .''}$$

50—51 Since also $udx + vdy$ must be a complete differential at any fixed time, let its integral be S, so that

(53) $$dS = udx + vdy + \frac{\partial S}{\partial t}dt .$$

Then

(54) $$\frac{\partial u}{\partial t}dx + \frac{\partial v}{\partial t}dy = \frac{\partial^2 S}{\partial x\,\partial t}dx + \frac{\partial^2 S}{\partial y \partial t}dy = d\frac{\partial S}{\partial t} .$$

Thus (52) becomes

(55) $$dp = \alpha dx - 2udu - 2vdv - 2d\frac{\partial S}{\partial t},$$

whence appears on integrating

(56) $$\begin{aligned} p &= \text{Const.} + \alpha x - u^2 - v^2 - 2\frac{\partial S}{\partial t}, \\ &= \text{Const.} + \alpha x - V^2 \quad - 2\frac{\partial S}{\partial t}. \end{aligned}$$

"If now we wish to take account of friction also, let us set it proportional to the pressure p. While the point l traverses the element ds, then the retarding force arising from friction $= p/f$," so that in place of (55) we have 53

(57) $$dp = \alpha dx - \frac{p}{f}ds - 2VdV - 2\frac{\partial S}{\partial t},$$

whence arises on integrating . . .

(58) $$p = \alpha x - V^2 - \frac{1}{f}e^{-\frac{s}{f}}\int e^{\frac{s}{f}}\left(\alpha x - V^2 - 2\frac{\partial S}{\partial t}\right)ds\ ."$$

In summary, the two formulae $udx + vdy$ and $udy - vdx$ must be complete differentials at the time t. 54

Turning now to the general case of motion in three dimensions, for the component accelerations [*i. e.*, accelerating forces] we obtain 55–56

(59) $$2\left(\frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v + \frac{\partial u}{\partial z}w + \frac{\partial u}{\partial t}\right)\left[= 2a_x\right], \quad etc.$$

Choosing the z-axis downward, set the accelerating force arising from gravity $= -1$. It would be superfluous to repeat the reasoning of §§ 44—54. We shall get 57

(60) $$\begin{aligned} \frac{\partial p}{\partial x} &= \quad\; -2\left(\frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v + \frac{\partial u}{\partial z}w + \frac{\partial u}{\partial t}\right)\left[= \quad\; -2a_x\right], \\ \frac{\partial p}{\partial y} &= \quad\; -2\left(\frac{\partial v}{\partial x}u + \frac{\partial v}{\partial y}v + \frac{\partial v}{\partial z}w + \frac{\partial v}{\partial t}\right)\left[= \quad\; -2a_y\right], \\ \frac{\partial p}{\partial z} &= -1 - 2\left(\frac{\partial w}{\partial x}u + \frac{\partial w}{\partial y}v + \frac{\partial w}{\partial z}w + \frac{\partial w}{\partial t}\right)\left[= -1 - 2a_z\right]. \end{aligned}$$

Since however the formula $dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz$ must be a complete differential, by forming $\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right)$, *etc.* we obtain [after some manipulation] 58–59

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)$$

(61)

$$+\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{\partial u}{\partial z}\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\frac{\partial w}{\partial x}=0\,,\quad \textit{etc.}$$

60 "It is now manifest that these three equations are satisfied by the following three values . . .

(62) $$\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}\,,\qquad \frac{\partial u}{\partial z}=\frac{\partial w}{\partial x}\,,\qquad \frac{\partial v}{\partial z}=\frac{\partial w}{\partial y}\,.$$

But these conditions are the same as those which are required in order that the formula $udx+vdy+wdz$ be a complete differential."

61—63 By analysis parallel to that of §§ 49—52 we obtain

(63) $$p=C-z-V^2-2\frac{\partial S}{\partial t}\,,$$

where

(64) $$dS=udx+vdy+wdz+\frac{\partial S}{\partial t}dt\,.$$

64—65 In summary, $udx+vdy+wdz$ must be exact, and $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$. "And to these two conditions the whole motion of fluids endowed with invariable density is subjected." The height p expressing the pressure is to be obtained from (63).

66 "Much more difficult however would be the question, if, the forces acting being given, along with the pressure in certain places, the motion of the fluid at the several points had to be determined." Then we should have to find functions u, v, w, and p satisfying both our equations and the specified conditions, "which work would certainly require the greatest force of calculation. It is fitting therefore to inquire in general into the nature of
67 functions proper to satisfy both criteria. Most conveniently therefore let us begin with that integral quantity S, whose differential $udx+vdy+wdz$ must be when the time is held constant." Since

(65) $$u=\frac{\partial S}{\partial x}\,,\qquad v=\frac{\partial S}{\partial y}\,,\qquad w=\frac{\partial S}{\partial z}\,,$$

substitution in (44) yields

(66) $$\frac{\partial^2 S}{\partial x^2}+\frac{\partial^2 S}{\partial y^2}+\frac{\partial^2 S}{\partial z^2}=0\,.$$

68 "Since it is not plain how this can be handled in general, I shall consider certain rather general cases. Therefore let

(67) $$S=(Ax+By+Cz)^n\,."$$

Substitution in (66) yields

(68) $$n(n-1)\,(Ax+By+Cz)^{n-2}\,(A^2+B^2+C^2)=0\,.$$

Thus $S = \text{const.}$ and $S = Ax + By + Cz$ are solutions for arbitrary A, B, C. But 69
if $n \neq 0, 1$, we must have

(69) $$A^2 + B^2 + C^2 = 0 \; .$$

Then (67) yields a solution, whatever is n, and moreover any sum of such solutions is
also a solution. By selecting such sums properly, we can get homogeneous polynomial 70
solutions [and the most general such solutions of degrees 0 through 4 are written out].
"Hence it is clear how these formulae are to be gotten for any order. First simply give 71
to the several terms the numerical coefficients which belong to them from the law of
permutation, or, equivalently, which arise when the trinomial $x + y + z$ is raised to
the power of that same order. Let indefinite letters A, B, C, *etc.*, be adjoined to the
numerical coefficients. Then, casting aside the numbers, observe whenever there occur
three terms of the type $LZx^2 + MZy^2 + NZz^2$ having a common factor Z formed from
the variables. Whenever this occurs, set the sum of the literal coefficients $L + M + N$
equal to zero." The case $n = 5$ is written down as an example of the method.

The case $S = A$ is a state of rest. Since A may be an arbitrary function of time, 72
by (63) it follows that so also may be p. Thus, *e. g.*, a fluid completely filling a closed
vessel remains in equilibrium even when subjected to arbitrary forces, since the pressure
varies accordingly with time. The case $S = Ax + By + Cz$ gives uniform motion. In 73
the case when $S =$ a polynomial of degree two, the different parts of the fluid are carried 74
in a varying motion. "Moreover a much greater variety can take place, if more elaborate
values are given to the function S." Now the uniform case is appropriate to the trans- 75
latory motion of a [rigid] solid body. It is possible to suspect that other rigid motions of
a fluid, such as a rotation, can occur. But such is not the case. For then the pyramid of
§ 26 would have to move so as to remain similar to itself. The condition of similarity yields 76

(70) $$\frac{\partial u}{\partial x} = 0 \, , \; \frac{\partial v}{\partial y} = 0 \, , \; \frac{\partial w}{\partial z} = 0 \, , \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \, , \; \frac{\partial v}{\partial z} = -\frac{\partial w}{\partial y} \, , \; \frac{\partial w}{\partial x} = -\frac{\partial u}{\partial z} \, ,$$

But these equations together with (62) show that the velocities u, v, w are constants. 77

"In order that the effect of the forces which act from the outside upon the fluid can 78
be ascertained, it is first necessary to determine those forces which are required for effecting
the motion which we have assumed to exist in the fluid. But indeed these are equivalent
to those forces which in fact work upon the fluid," which we have found in § 56. For a
fluid element whose volume or mass is $dx\,dy\,dz$, the "moving forces required for the
motion" will be

(71) $$2\,dx\,dy\,dz\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\right)\left[= 2a_x\,dx\,dy\,dz\right] \, , \; \textit{etc.},$$

"whence by triple integration the components of the total forces which must act on the
whole mass of fluid may be obtained." Using (62) and putting 79—80

(72) $$T = u^2 + v^2 + w^2 + 2\frac{\partial S}{\partial t} \; ,$$

we get

(73) $$[2a_x =]\, 2\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\right] = \frac{\partial T}{\partial x}\,, \quad \textit{etc.},$$

"and by triple integration these formulae are to be extended through the whole mass of the fluid, so that thence forces equivalent to all [the pressure forces] and their mean directions may be obtained. Truly this discussion is for a deeper investigation, on which I do not pause here."

81 But by means of (72) we get from (63) a simpler expression for the pressure:

(74) $$p = C - z - T\,.$$

"But if moreover an arbitrary particle is acted upon by an accelerating force whose components are Q, q, φ, after a calculation like that above has been carried out the pressure will be found by

(75) $$p = C + \int (Q\,dx + q\,dy + \varphi\,dz) - T\,,$$

whence is plain that the differential $Q\,dx + q\,dy + \varphi\,dz$ must be complete, as besides a state of equilibrium, or at least a possible one, could not exist. That this condition must be imposed on the acting forces Q, q, φ was shown very clearly by the famous Mr. Clairaut.

82 "Behold therefore the principles of the entire doctrine of the motion of fluids, which, even if they at first sight may seem insufficiently fruitful, nevertheless embrace almost everything yet treated both in hydrostatics and in hydraulics, so that these principles must be regarded as having very broad extent. In order that this shall appear more clearly, it is worthwhile to show how the precepts learned in hydrostatics and hydraulics follow . . ."

83 To consider equilibrium, put $u = v = w = 0$. Then (75) yields

(76) $$p = C + \int (Q\,dx + q\,dy + \varphi\,dz)\,,$$

84 where C is a function of time. Since p is a function of position only, at any given time, the differential $Q\,dx + q\,dy + \varphi\,dz$ must be complete, say

(77) $$dP = Q\,dx + q\,dy + \varphi\,dz\,.$$

Then (76) becomes

(78) $$p = C + P\,.$$

85, 86 For the case of gravity, $p = C - z$. Both P and C may depend on time. On a boundary where the fluid is subjected to no force, $p = 0$. Hence the equation of this boundary is $P = \text{const.} = E$, so that (78) becomes in general $p = P - E$.

87 "Next, everything which has hitherto been brought out concerning the motion of a fluid through tubes is easily derived from these principles. The tubes are usually regarded as very narrow, or else assumed to be such that through any section normal to the tube the fluid flows across with equal motion: whence originates this rule, that the speed of the

fluid at any place in the tube is reciprocally proportional to the amplitude." Let $y = y(x)$, $z = z(x)$ be equations for the tube. Let the amplitude be r^2, so that if ȣ and f^2 be the 88
speed and amplitude at a fixed point, then $V = f^2 \text{ȣ}/r^2$. Hence

(79) $$u = \frac{v}{y'} = \frac{w}{z'} = \frac{f^2 \text{ȣ}}{r^2} \cdot \frac{1}{\sqrt{1 + y'^2 + z'^2}} .$$

Again use (64): 89

(80) $$dS = \frac{f^2 \text{ȣ}}{r^2} dx \sqrt{1 + y'^2 + z'^2} = \frac{f^2 \text{ȣ}}{r^2} ds ,$$

so that

(81) $$S = \text{ȣ} \int \frac{f^2 ds}{r^2} .$$

To calculate the pressure, by (72) we obtain 90

(82) $$T = \frac{f^4 \text{ȣ}^2}{r^4} + 2 \frac{d\text{ȣ}}{dt} \int \frac{f^2 ds}{r^2} .$$

Then by (75) follows

(83) $$p = C + \int \left(Q\,dx + q\,dy + \varphi\,dz \right) - \frac{f^4 \text{ȣ}^2}{r^4} - 2 \frac{d\text{ȣ}}{dt} \int \frac{f^2 ds}{r^2} ,$$

"which is that same formula which is commonly brought out for the motion of a fluid through tubes; but now much more widely valid, since arbitrary forces acting on the fluid are assumed here, while commonly this formula is restricted to gravity alone."

***Part XB.* Comments on the paper summarized above.**

This paper is famous for its derivations of the continuity equation (44) and the dynamical equations (60) for ideal incompressible fluids. Here for the first time the *kinematical* and the *dynamical* aspects of the theory of continua are carefully separated. In the writings of the BERNOULLIS and D'ALEMBERT no such separation is to be found, although we have seen a hint of it in a letter from JOHN BERNOULLI to EULER (see above, p. XXXIV). While clear and compelling in contrast to all that had been done before in hydrodynamics, this memoir does not attain the magnificent clarity of EULER's best writing. On the other hand, it is taut with important new ideas, many of which are to be worked out in later papers.

The derivation of the continuity equation is similar to one of those used by D'ALEMBERT in special cases (above, pp. LIII and LVI), but with an interesting difference. EULER supposes (x, y, z) is carried into $(x + u\,dt, \ldots)$, but in the calculation he retains terms $O(dt^2)$. The result cannot yield a formula for the volume change correct in general beyond $O(dt)$. However, after giving in § 15 of his later paper E 226 (see below, p. LXXXIV), a derivation in which all terms $O(dt^2)$ are systematically neglected, EULER wrote: "If one still has any doubt about the justness of this conclusion, one has only to read my Latin work, *Principia motus fluidorum*, where I have calculated this volume without neglecting anything." Now in fact the calculation here is correct if and only if the terms

$O(dt^2)$ in the displacement are actually zero; equivalently, if the velocity field is linear. The interesting invariant form of (43) suggests that it is worthwhile to follow out this idea. Elsewhere I have done so[1]). Let $\boldsymbol{A}$ be any $n \times n$ matrix, and consider the one parameter family of transformations of n-space whose matrices are $\boldsymbol{A}t + \boldsymbol{I}$, $\boldsymbol{I}$ being the unit matrix. Adopt the EUCLIDEAN definition of volume of a paralleletope with one vertex at 0, *viz.* $V \equiv \det \boldsymbol{x}_{(i)}$, $\boldsymbol{x}_{(i)}$ being the vectors to the remaining n vertices. Then it can be shown that for any such paralleletope

$$(84) \qquad \frac{V(t)}{V(0)} = \det | \boldsymbol{A}t + \boldsymbol{I} | = \sum_{r=0}^{n} t^r I_r \,,$$

where I_r is the sum of the r-rowed principal minors of $\boldsymbol{A}$. This formula gives a geometrical interpretation for the expansion of the secular determinant. EULER's results (41) and (43) are the special cases $n = 2,3$ of (84). Thus EULER is the discoverer of the secular expansion. Moreover, from (84) follows

$$(85) \qquad \left.\frac{1}{r!\,V}\frac{d^r V}{dt^r}\right|_{t=0} = \begin{cases} I_r\,, & r = 1, 2, \ldots, n\,, \\ 0\,, & r \geqq n+1\,, \end{cases}$$

yielding a geometrical interpretation for the invariant coefficients I_r, for which I have found no earlier occurrence.

In the dynamical part the pressure p is introduced, as in the paper on rivers (above, p. LXV), as mechanically equivalent to the forces of mutual action. The explanation in § 43 is not very illuminating, however; a somewhat better one is contained in the summary; and in later papers EULER improved it. When he writes in § 49 that it is absolutely necessary to calculate the pressure, perhaps he is criticizing D'ALEMBERT.

EULER's entire viewpoint toward fluid dynamics is characterized and distinguished from earlier attempts by careful use of preliminary *kinematical* analysis. EULER's formulae (46) and (59) generalize D'ALEMBERT's (23), but for the first time it is stated that it is indeed the acceleration components which are being found.

As soon as EULER establishes the dynamical equations, he derives the condition of integrability for the pressure. The result, in the two forms (50) and (61), may be written in vector notation as follows:

$$(86) \qquad \frac{D\boldsymbol{W}}{Dt} = -\boldsymbol{W} \operatorname{div} \boldsymbol{v}\,,$$

$$(87) \qquad \frac{D\boldsymbol{W}}{Dt} = \boldsymbol{W} \cdot \operatorname{grad} \boldsymbol{v} - \boldsymbol{W} \operatorname{div} \boldsymbol{v}\,,$$

where $\boldsymbol{W}$ is the vorticity vector: $\boldsymbol{W} \equiv \operatorname{curl} \boldsymbol{v}$. The former, which generalizes D'ALEMBERT's result (25) for steady flow, is usually regarded as a discovery of STOKES, and the latter is usually called "HELMHOLTZ's vorticity equation." EULER did not introduce a symbol for

1) "*Generalization of a geometrical theorem of* EULER," Comm. mat. Helv. 27, 233—234 (1953).

the vorticity or construct a kinematical interpretation for it. HELMHOLTZ's treatment is greatly superior from the viewpoints of kinematics and mechanics, but D'ALEMBERT and EULER brought out more distinctly the purely analytical nature of these equations as integrability conditions. Quite erroneously, however, both stated that these equations *require* $W = 0$, *i. e.*, in modern terms, that the motion shall be irrotational. Perhaps EULER was indeed influenced by D'ALEMBERT, since it was D'ALEMBERT who originated this error (see above, p. LIV) both in statement and in method of proof. In later papers, as we shall see, EULER took pains to emphasize that on the contrary $W = 0$ yields only a very special class of solutions.

With the aid of the foregoing false step EULER is able in § 54 to repeat D'ALEMBERT's summary of the whole theory in the plane case as a statement that $udx + vdy$ and $udy - vdx$ be complete differentials. Another false conclusion results in § 81, where EULER apparently claims to prove that only force fields under whose action equilibrium is possible can occur in nature. The analysis there is correct, however, and proves in fact that a necessary condition for potential flow in a homogeneous incompressible fluid is the completeness of the differential $Qdx + qdy + \varphi dz$. [EULER's conclusion that a fluid cannot flow irrotationally when subject to forces under which it could not remain at rest is not quite right: forces derivable from a many-valued potential may produce potential flow in a multiply-connected region, although they are not compatible with rest, since the cyclic term in the force potential may cancel that in the acceleration-potential, thus yielding a single-valued pressure field for the fluid in motion, but no possibility of equilibrium. In fact, suppose $\int (Qdx + qdy + \varphi dz) = v$, where v is a particular determination of a cyclic function; in (72) put $S = \frac{1}{2} tv + S' + S''$, where S' is single-valued and S'' is steady but possibly cyclic; then the velocity is single-valued, but the acceleration-potential T is the sum of v and a single-valued function; so that p as given by (75) is single-valued, whatever are S' and S''. For *steady* potential flow, however, it is obvious that v must be single-valued, as for equilibrium.]

The condition of irrotational motion enables EULER to introduce by (53) and (64) the function S, which HELMHOLTZ was to call the *velocity-potential*. In terms of it EULER quickly obtains the "BERNOULLI equation" for unsteady potential flow in the forms (56) and (63). The result (40) in the paper on rivers indeed contains (56) as a special case, provided the flow is steady, but there is no indication there that the function $\int wdz$ will reduce to a constant in a potential flow; indeed, the whole matter is now confused badly by the conclusion that potential flow is *necessary*.

In § 53 EULER generalizes his earlier theory of friction in tubes (see above, pp. XLVIII —L). From (57), however, partial differential equations do not follow, since ds is a differential along the particle path, whose direction is not known *a priori*. Since a velocity potential S has already been assumed to exist, however, the *kinematics* of the flow is unchanged by friction. Once the motion is known, the result (58) then gives a method of calculating the loss of head due to friction on each trajectory. EULER's theory is thus both consistent and simple for potential flows. Of course we know now that friction has a more far reaching effect on the local flow quantities than EULER here assumes.

Turning to the consequences of the existence of a velocity-potential in three dimensions, EULER sees at once that S must satisfy the potential equation (66) ("LAPLACE's equation"). Not perceiving a general solution, he seeks special ones of polynomial form. The analysis in §§ 68—69 may be interpreted as showing that $f(z + ix \cos u + iy \sin u)$ is a harmonic function of x, y, z. Starting from this observation and from a series of the type given in § 69, E. T. WHITTAKER[1]) was to be able to construct the general solution of the potential equation. In § 71 EULER gives a rule for finding the most general homogeneous polynomial harmonic of degree n. A more compact statement of this same rule has been obtained by WHITTAKER.

Noting that the harmonic of degree 1 represents a state of uniform motion, EULER then asks if other rigid fluid motions are possible. To approach this problem, he seeks a criterion for locally and instantaneously rigid motion. The results (70), stating that the symmetric part of the velocity gradient must vanish, are now generally called "KILLING's equations." EULER thence draws a conclusion (§ 77) which can be stated as follows: the only rigid potential motion is a state of uniform translation[2]).

To determine the resultant force on a fluid mass, EULER introduces the *acceleration-potential* T, given by (73) and expressed kinematically through (72). In terms of the acceleration vector $\boldsymbol{a}$, we may write the result (in general units) as

$$\text{(88)} \qquad \boldsymbol{a} = \operatorname{grad}\left[\frac{\partial S}{\partial t} + \tfrac{1}{2} V^2\right] = \operatorname{grad}\left[\frac{\partial S}{\partial t} + \tfrac{1}{2} (\operatorname{grad} S)^2\right],$$

a formula which was to reappear in the work of VESSIOT[3]). The acceleration-potential is then expressed dynamically by (74). If we carry out the integration of the expression (71), as EULER recommends, by use of GREEN's theorem we obtain for the resultant force (of inertia only, and in general units),

$$\text{(89)} \qquad \boldsymbol{F} = \varrho \int \boldsymbol{a}\, dv = \varrho \oint T d\boldsymbol{S} = - \oint (p + \varrho g z) d\boldsymbol{S}.$$

In (77) we see for the first time the differential equation of hydrostatics (for earlier statements of the principles of hydrostatics, see Parts I and II of this introduction).

The theory of flow in narrow tubes given in §§ 87-90 is the first attempt at a really general treatment. The continuity condition $V = f^2 \, \text{ɣ} / r^2$, as in the older works, is set down as a principle, not derived from (44). The dynamical equation (83), however, follows from the general "BERNOULLI equation" (75), since the acceleration-potential T has first been determined from the kinematics of the flow. These last brilliant paragraphs present the hydraulic theory with a conciseness and elegance not only unattained by previous writers but also not imitated by subsequent ones.

1) *"On the partial differential equations of mathematical physics,"* Math. Ann. 57, 333—355 (1903).

2) Professor KUERTI has remarked that since EULER claimed to prove in §§ 46—48, 58—60 that *all* fluid motions are potential flows, it now must follow that a fluid cannot rotate as a rigid body! It is strange that neither D'ALEMBERT nor EULER noticed that figures of relative equilibrium furnish counter-examples to their assertion that potential flow is necessary.

3) *"Sur les transformations infinitésimales et la cinématique des milieux continus,"* Bull. sci. math. (2) 35^1, 233—244 (1911).

The analytical complications of the subject prevented EULER from attaining the great aims set out in the summary. Of course, the program stated in the first sentence remains to this day the program only, not the fruit of the theory. The formidable mathematical difficulties, now all too well known, were first encountered by EULER in this paper and the preceding, in which and in D'ALEMBERT's *Essay* we find the first dim conception of a field theory governed by partial differential equations, whose integrals are to be adjusted in conformity to prescribed initial conditions and boundaries (§ 66). "But nevertheless the whole theory has been reduced to pure analysis, and what remains to be completed in it depends solely upon subsequent progress in analysis."

We come next to a sequence of three papers, written in 1753—1755, which constitute an elegant, organized, and complete treatise on the whole mathematical theory of fluid mechanics as it stood after EULER's first researches.

Part XIA. **Contents of the *General principles of the state of equilibrium of fluids*** (E 225, pp. 2—53) (1753)

(*Part 1. General principles.*) "I propose here to develop the principles on which all 1
of hydrostatics, or the science of the equilibrium of fluids, is founded. To give them the greatest extent of which they are susceptible, I shall include in my researches not only fluids such as water and the other liquids, which have everywhere the same degree of density, and of which it is said that they suffer no compression; but also those fluids which are composed of particles of varying density, whether this difference befalls them in virtue of their own nature, or results from the forces with which the particles mutually press one another. It is plain that to this latter type must be relegated the air and other fluid bodies which are called elastic. Beyond this I shall not limit my researches to the case of gravity as the only force, but I shall extend them to arbitrary forces acting upon each particle of the fluid.

"There is the program which I propose to execute, whence it is immediately clear 2
that the common principles of hydrostatics . . . are only a very particular case of those which I am going to establish here." Even though the equilibrium of elastic fluids has been studied, "the principles which have been established for them seem so different from [those for incompressible fluids] that one could hardly trace them back to a common origin, founded in the general nature of fluids.

"Although I envisage here such a great generality, both in respect to the nature of 3
the fluid as well as the forces which act upon each of its particles, I fear not at all the reproaches often justly directed at those who have undertaken to bring to a greater generality the researches of others. I agree that too great a generality often obscures rather than enlightens, leading sometimes to calculations so entangled that it is extremely difficult to deduce their consequences in the simplest cases . . . But in the subject I propose 4
to explain, the very reverse occurs: the generality which I undertake, rather than dazzling our lights, will the more discover to us the true laws of Nature in all their brilliance, and we shall find therein even stronger reasons to admire her beauty and simplicity. It will be an important lesson to learn that principles which had been thought connected to